When blowups are flat

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Abstract
A blowup is often given as the standard example of a non-flat morphism. On the other hand, if we have a flat morphism from $X$ to $Y$ and we make a blowup of $X$, it can often happen that the resulting composite morphism to $Y$ remains flat. In this brief note, we give a simple condition in terms of dimensions.

1 Introduction
Suppose we have a flat morphism $\pi: X \to Y$, and a closed subscheme $Z$ of $X$ whose complement is $Y$-dense in $X$. Write $\tilde{\pi}: \tilde{X} \to Y$ for the map from the blowup of $X$ along $Z$ to $Y$. Under suitable non-singularity hypotheses, we prove that $\tilde{\pi}$ is flat if and only if $\dim Z \geq \dim Y - 1$. The proof is simple and uses standard techniques; probably it is well-known, but I could not find it in the literature.

2 The main result
Let $i: Z \hookrightarrow X$ be an lci closed immersion with $X$ Cohen-Macaulay, and $\pi: X \to Y$ be a flat finite-type morphism to a regular scheme, with all fibres equidimensional. Let $\tilde{X} \to X$ be the blowup of $X$ along $Z$. We are interested in when the composite morphism $\tilde{\pi}: \tilde{X} \to Y$ is flat.

Let $p \in \tilde{X}$ be a closed point, and write $x$ for its image in $X$ and $y$ for its image in $Y$.

Theorem 2.1. Assume that the complement of $Z_y$ is dense in the fibre $X_y$. Then the morphism $\tilde{\pi}$ is flat at $p$ if and only if $\dim_x Z \geq \dim_y Y - 1$.

Proof. Write $E \to \tilde{X}$ for the exceptional locus (i.e. the preimage of $Z$). Since the ideal sheaf $I$ of $Z$ is locally generated by a regular sequence, the sheaf of graded
rings \( i^*(\bigoplus_i \mathcal{I}^i / \mathcal{I}^{i+1}) \) is locally a polynomial ring over \( Z \) in \( \dim X - \dim Z \) variables, so \( E \) is a \( \mathbb{P}^r \) bundle over \( Z \) where \( r = \dim X - \dim Z - 1 \). Since \( X \) is Cohen-Macaulay and \( Z \) is lci we see that \( Z \) is also Cohen-Macaulay, and hence the same is true for \( E \). Moreover, \( \hat{X} \setminus E = X \setminus Z \) is clearly Cohen-Macaulay, and \( E \) is a Cartier divisor in \( \hat{X} \) and is Cohen-Macaulay, hence \( \hat{X} \) itself is Cohen-Macaulay (this part of the argument is based on a MathOverflow answer of Sándor Kovács).

From the above description of \( E \) we see that

\[
\dim_p E_x = \dim_x X - \dim_x Z - 1.
\]

Flatness of \( \pi \) tells us that \( \dim X_y = \dim X - \dim Y \), so

\[
\dim X \geq \dim Y - 1 \iff \dim_p E_x \leq \dim_x X_y.
\]

Now (using the \( Y \)-density) we apply Lemma 2.2 with \( T = X_y \) and \( S = \hat{X}_y \) (perhaps after shrinking to affine patches) to see that

\[
\dim_p E_x \leq \dim X_y \iff \dim_x X_y = \dim_p \hat{X}_y.
\]

Now \( \dim_p \hat{X} = \dim_x X \), and flatness of \( \pi \) tells us that

\[
\dim X_y = \dim X - \dim Y,
\]

so combining we find that

\[
\dim X_y = \dim_p \hat{X}_y \iff \dim_p \hat{X}_y = \dim_p \hat{X} - \dim Y.
\]

Since \( \hat{X} \) is Cohen-Macaulay, we can apply the Miracle Flatness Theorem [Mat86, Theorem 23.1] to see that flatness of \( \hat{\pi} \) is equivalent to this last equality.

**Lemma 2.2.** Let \( k \) be a field, and \( T/k \) a scheme of finite type, with all irreducible components of \( T \) having the same dimension. Let \( f: S \to T \) be a proper morphism which is an isomorphism over some dense open \( U \subseteq T \). Let \( B := f^{-1}(T \setminus U) \), a closed subset of \( T \). Assume \( B \) is irreducible. Then

\[
\dim B \leq \dim T \iff \dim S = \dim T
\]

**Proof.** By [Liu02, 2.5.19] we know \( \dim U = \dim T \), so \( \dim S \geq \dim T \). If \( \dim B > \dim T \) then \( \dim S > \dim T \), so one implication is clear. So let us assume that \( \dim B \leq \dim T \), and try to show that \( \dim S = \dim T \).

Let \( b \in B \) be a closed point. We will show that \( \dim_b S \leq \dim T \). We consider two cases:

**Case 1:** \( B \) is a maximal irreducible subset of \( S \).

Then \( \dim_b S = \dim_b B \) and we are done.
Case 2: There exists $B \subseteq V \subseteq S$ with $V$ irreducible. Then $V \cap f^{-1}U$ is non-empty, so $\dim_b V = \dim V = \dim f^{-1}U \cap V$. Let $\eta$ be the generic point of $V$. Then $\dim_\eta S = \dim_\eta T$ since the local rings are isomorphic. And since our schemes are of finite type over $k$ we get that

$$\dim_b S = \dim_b V + \dim_\eta S$$

and

$$\dim T = \dim f^{-1}U \cap V + \dim_\eta T,$$

which yields the desired equality.

2.1 Examples

Note that if $\dim Y \leq 1$ then flatness of $\tilde{\pi}$ always holds; indeed, being flat over a Dedekind ring is equivalent to being torsion free, so this is no surprise. If $\dim Y = 2$ then such a blowup will preserve flatness if and only if $\dim Z \geq 1$; in other words, one breaks flatness by blowing up at a point, but blowing up along a non-vertical curve is safe.

One can easily apply this result to see when the blowup along a diagonal remains flat. Let $Y$ be a regular scheme and let $T/Y$ be smooth. Let $X = T^n$ for some $n \geq 2$ (fibre product over $Y$), and let $Z \cong T$ be the small diagonal. Then the blowup $\tilde{X}$ of $X$ along $Z$ is flat over $Y$ if and only if $\dim T \geq \dim Y - 1$.

References
