When blowups are flat

David Holmes

May 25, 2017

Abstract

A blowup is often given as the standard example of a non-flat morphism. On the other hand, if we have a flat morphism from X to Y and we make a blowup of X, it can often happen that the resulting composite morphism to Y remains flat. In this brief note, we give a simple condition in terms of dimensions.

1 Introduction

Suppose we have a flat morphism $\pi\colon X \to Y$, and a closed subscheme Z of X whose complement is Y-dense in X. Write $\tilde{\pi}\colon \tilde{X} \to Y$ for the map from the blowup of X along Z to Y. Under suitable non-singularity hypotheses, we prove that $\tilde{\pi}$ is flat if and only if $\dim Z \geq \dim Y - 1$. The proof is simple and uses standard techniques; probably it is well-known, but I could not find it in the literature.

2 The main result

Let $i\colon Z \rightarrowtail X$ be an lci closed immersion with X Cohen-Macaulay, and $\pi\colon X \to Y$ be a flat finite-type morphism to a regular scheme, with all fibres equidimensional. Let $\tilde{X} \to X$ be the blowup of X along Z. We are interested in when the composite morphism $\tilde{\pi}\colon \tilde{X} \to Y$ is flat.

Let $p \in \tilde{X}$ be a closed point, and write x for its image in X and y for its image in Y.

Theorem 2.1. *Assume that the complement of Z_y is dense in the fibre X_y. Then the morphism $\tilde{\pi}$ is flat at p if and only if* $\dim_x Z \geq \dim_y Y - 1$.

Proof. Write $E \rightarrowtail \tilde{X}$ for the exceptional locus (i.e. the preimage of Z). Since the ideal sheaf $\mathscr{I}$ of Z is locally generated by a regular sequence, the sheaf of graded rings $i^*\left(\bigoplus_i \mathscr{I}^i/\mathscr{I}^{i+1}\right)$ is locally a polynomial ring over Z in $\dim_x X - \dim_x Z$ variables, so E is a $\mathbb{P}^r$ bundle over Z where $r = \dim_x X - \dim_x Z - 1$. Since X is Cohen-Macaulay and Z is lci we see that Z is also Cohen-Macaulay, and hence the same is true for E. Moreover, $\tilde{X} \setminus E = X \setminus Z$ is clearly Cohen-Macaulay, and E is a Cartier divisor in $\tilde{X}$ and is Cohen-Macaulay, hence $\tilde{X}$ itself is Cohen-Macaulay (this part of the argument is based on a mathoverflow answer of Sándor Kovács).

From the above description of E we see that

$$\dim_p E_x = \dim_x X - \dim_x Z - 1.$$

Flatness of π tells us that $\dim_x X_y = \dim_x X - \dim_y Y$, so

$$\dim_x Z \geq \dim_y Y - 1 \iff \dim_p E_x \leq \dim_x X_y.$$

Now (using the Y-density) we apply lemma 2.2 with $T = X_y$ and $S = \tilde{X}_y$ (perhaps after shrinking to affine patches) to see that

$$\dim_p E_x \leq \dim_x X_y \iff \dim_x X_y = \dim_p \tilde{X}_y.$$

Now $\dim_p \tilde{X} = \dim_x X$, and flatness of π tells us that

$$\dim_x X_y = \dim_x X - \dim_y Y,$$

so combining we find that

$$\dim_x X_y = \dim_p \tilde{X}_y \iff \dim_p \tilde{X}_y = \dim_p \tilde{X} - \dim_y Y.$$

Since $\tilde{X}$ is Cohen-Macaulay, we can apply the Miracle Flatness Theorem [Mat86, Theorem 23.1] to see that flatness of $\tilde{\pi}$ is equivalent to this last equality. □

Lemma 2.2. *Let k be a field, and T/k a scheme of finite type, with all irreducible components of T having the same dimension. Let $f\colon S \to T$ be a proper morphism which is an isomorphism over some dense open $U \subseteq T$. Let $B := f^{-1}(T \setminus U)$, a closed subset of T. Assume B is irreducible. Then*

$$\dim B \leq \dim T \iff \dim S = \dim T$$

Proof. By [Liu02, 2.5.19] we know $\dim U = \dim T$, so $\dim S \geq \dim T$. If $\dim B > \dim T$ then $\dim S > \dim T$, so one implication is clear. So let us assume that $\dim B \leq \dim T$, and try to show that $\dim S = \dim T$.

Let $b \in B$ be a closed point. We will show that $\dim_b S \leq \dim T$. We consider two cases:

Case 1: B is a maximal irreducible subset of S.
Then $\dim_b S = \dim_b B$ and we are done.

Case 2: There exists $B \subsetneq V \subseteq S$ with V irreducible.
Then $V \cap f^{-1}U$ is non-empty, so $\dim_b V = \dim V = \dim f^{-1}U \cap V$. Let η be the generic point of V. Then $\dim_\eta S = \dim_\eta T$ since the local rings are isomorphic. And since our schemes are of finite type over k we get that

$$\dim_b S = \dim_b V + \dim_\eta S$$

and

$$\dim T = \dim f^{-1}U \cap V + \dim_\eta T,$$

which yields the desired equality. □

2.1 Examples

Note that if $\dim Y \leq 1$ then flatness of $\tilde{\pi}$ always holds; indeed, being flat over a Dedekind ring is equivalent to being torsion free, so this is no surprise. If $\dim Y = 2$ then such a blowup will preserve flatness if and only if $\dim Z \geq 1$; in other words, one breaks flatness by blowing up at a point, but blowing up along a non-vertical curve is safe.

One can easily apply this result to see when the blowup along a diagonal remains flat. Let Y be a regular scheme and let T/Y be smooth. Let $X = T^n$ for some $n \geq 2$ (fibre product over Y), and let $Z \cong T$ be the small diagonal. Then the blowup $\tilde{X}$ of X along Z is flat over Y if and only if $\dim T \geq \dim Y - 1$.

References

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Mat86] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.