Degenerations of Jacobians of algebraic curves. I

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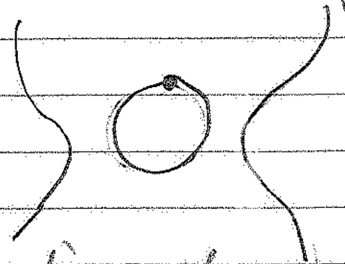
§ ~~1~~ Some examples

over a field $\mathbb{Q}$

1) Let $E \subseteq \mathbb{P}^2_{\mathbb{Q}}(1,1,2)$ be defined by $y^2 = x^4 - 2x^2 + \frac{1}{4}$



$x=0,\ y=\frac{1}{2}, z=1$ (say)

~~We can mark a pt.~~

~~This has a natural marked pt at infinity (0:1:0).~~

· There is a unique algebraic gp. law on E for which ~~(0:1:0)~~ $(0,\frac{1}{2})$ acts as the identity.

$$\left(\begin{array}{l} m: E \times E \to E \\ i: E \to E \\ e: \operatorname{Spec}\mathbb{Q} \to E \quad \text{s.t.} \ldots \\ e \mapsto (0:1:0) \end{array}\right).$$

2) Over a 1-d base. Let $S = \mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec}\mathbb{Q}[t]$

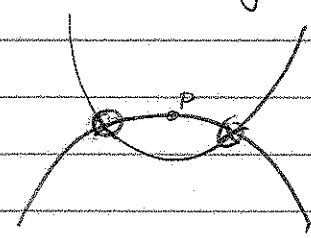
$$E: y^2 = x^4 - 2x^2 + (t-1)^2 \subseteq \mathbb{P}^2_S(1,1,2)$$

· ~~Has~~ Choose a section $x=0$, $y=t-1$ (say).

· Away from $t=0,1,2$, the curve E_t is smooth, pos. genus, has marked pt, so has canonical algebraic gp. law.

What happens to the group law as $t \to 0$?
It turns out that it degenerates to a group law on the smooth locus of the fibre ~~$E_0(t=0)$~~: ~~[illegible]~~

$$E_{t=0}: \; y^2 = (x^2-1)^2.$$



Case 3: over a base of dimension 2

$$S = \cancel{\operatorname{Spec} \mathbb{C}[[s,t]]} \cong_{\text{within}} \mathbb{A}^2_{\mathbb{C}}.$$

$$E: \; y^2 = (x^2-1)\left(x^2 + x - (s-1)^2 + tx - \frac{(s-1)^2}{t+1}x\right). \quad \subset \mathbb{P}_S(1,1,2)$$

$\sim$: $y^2 = (x-1)(x-1-s)(x+1)(x+1+t)$, ~~étale quasi~~ section ~~through~~ above $x=0$: $y = \sqrt{(1+s)(1+t)}$ locally near $s=t=0$

· ~~Choose a section $x=0$, $y=s-1$.~~

· For 'most' $(s,t) \in S$, we have that the fibre $E_{s,t}$ is smooth, proper, $g=1$ & section $\rightsquigarrow$ canonical group law.

· Fibre over $s=t=0$ is again given by ~~$E_{0,0}$~~:

$$E_{s=t=0}: \; y^2 = (x^2-1)^2.$$

· However, ~~there is NO canonical g~~ the group law on the smooth fibres does NOT extend to a group law on the smooth locus of $E_{s=0,t=0}$.

· ~~The~~ One way to see this: restricting to a line $t = \lambda \cdot s$ in the base S gives a 1-parameter family, & the gp law extends to $E^{sm}_{0,0}$ as in case 2. However, this group law on $E^{sm}_{0,0}$ depends continuously on λ!

~~Last~~ (3)

say Summarising:

- Over a field, get a canonical gp law.

- 1-parameter ~~generating~~ family: get a canonical gp law on every smooth fibre, & these degenerate to a canonical gp law on smooth locus of singular fibre;

- 2-parameter family: canonical gp laws on the smooth fibres, but this does <u>not</u> degenerate to a gp law on sm. locus of singular fibres - the gp law there depends on the 'direction' in which you degenerate to it.

Some (vague) questions:

- In case 3, there is no 'good' gp law on $E_{0,0}^{sm}$. Is there any 'nice' object which the group laws on smooth fibres do degenerate to ~~E_{00}~~ (some kind of 'Neron model'?)

write: 'is there a good limit as $s,t \to 0$?'

- The group law we get on E_{00}^{sm} depends on choice of line $s=\lambda t$ through $(0,0)$. Maybe things behave better after we blow up $(0,0)$?

write: 'does blowing up S help?'

- Is this bad behaviour over $A^2_{\mathbb{C}}$ 'typical', or did we just make an unfortunate choice of equation for E?

write: 'Is this typical behaviour?'

In order to make these questions precise (& to answer them), need some definitions.

<u>Def</u>: let S a scheme, & C/S a smooth, projective curve w. connected geometric fibres. The ~~Ja~~ jacobian Jac_C of C is an abelian scheme over S (= proper smooth gp scheme ~~conn~~ w. conn. fibres).
Eg. define it as
$Pic^0_{C/S}$ ~~in étale topology~~

$Pic^0_{C/S}$ = (fibrewise-conn-comp of id of | étale sheaf assoc. to) $Sch_S \longrightarrow Ab$

$T \longmapsto Pic(C \times_S T)$.

<u>Def</u>: let S a regular scheme, $U \subseteq S$ ~~sch th~~ dense open, A/U abelian.

A <u>Néron model</u> for A/S is a smooth separated gp alg. space N/S ~~st~~ w. a smap ~~A~~ ~~A~~ $A \hookrightarrow N$ s.t.

$\forall$ smooth morphisms $T \to S$ of alg. spaces, & ~~morphs~~ U-morphisms $f : T_U \to A$, $\exists !$ S-morphism $F : T \to N$ s.t. $F|_{T_U} = f$.

<u>Thm</u> [Néron, 1965]. If $\dim S = 1$ & S regular, $U \subseteq S$ d. open then every ab. sch. A/U admits a NM over S.

This is what was underlying ~~ove~~ the nice behaviour in our 1-d examples.

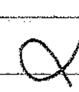
~~In order to make these questions precise (& to answer them) we will need some (more) definitions.~~

Def: • Let k a sep. cl. field. A semistable curve over k is a proper morphism $f: C \to \operatorname{Spec} k$ s.t.

- C connected & reduced;
- every irred comp of C has dimension 1;
- only non-smooth pts are 'ordinary double pts'

(i.e. completed local ring $\simeq$ to $\dfrac{k[\![x,y]\!]}{xy}$).

• Let S a scheme. A semistable curve over S is a proper flat morphism $f: C \to S$, whose fibres over geom. pts are semistable curves in the above sense.

eg nodal curve $y^2 = x^2(x-1)$  ✓

eg NOT cusp $y^2 = x^3$  ✗

Now in a position to ask precise questions:

Let S a reg. scheme, $U \subseteq S$ dense open, C/S s-stable & sm. $/U$

let $J = \mathrm{Jac}_{C_U/U}$

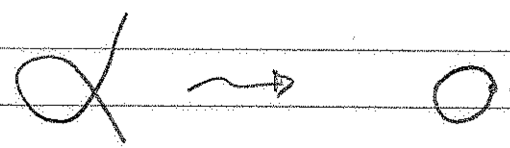
- Does J admit a NM over S? If not, how often?
- If not, does it do so after eg. blowing up S?

To answer these Qs, need def'n of 'aligned curve'.

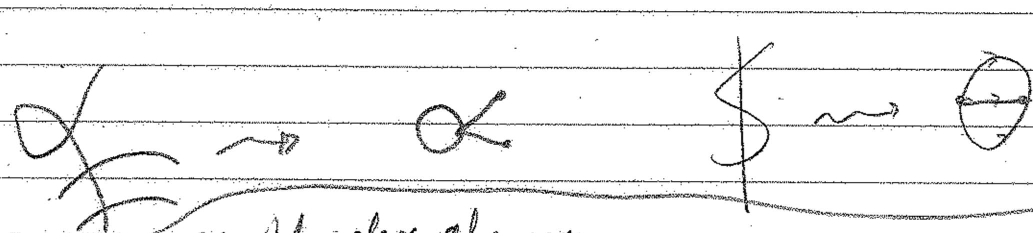
Def: Let C/k os-stable curve over a sep. cl. field. The ~~reduction~~ dual graph of C has

- 1 vertex for each irred. comp of C.
- 1 edge for each sing pt; edge goes between the two irred comps containing it.
- ~~Given vertices V_1, V_2, there is an edge joining V_1 to V_2 for each non-sm. pt of C at which V_1 meets V_2~~

eg:



say what doing or write them



smooth cplex alg. var

~~stably manifold~~

lemma ~~Let S a loc. Noeth. scheme,~~ & $C \to S$ a ~~semi-stable curve.~~ Let c a ~~geometric pt~~ non-smooth pt of C lying over ~~geom pt~~ pt s of S. Then $\exists\, \alpha \in \mathfrak{m}_{\mathcal{O}^{et}_{S,s}}$ & an isomorphism of complete local rings

$$\frac{\hat{\mathcal{O}}^{et}_{S,s}[[u,v]]}{(uv-\alpha)} \cong \hat{\mathcal{O}}^{et}_{C,c}.$$

Moreover, this α is unique up to mult by units in R; we call $\alpha\,\mathcal{O}^{et}_{S,s}$ the singular ideal of c.

[illegible] in places

or: $S/\mathbb{C}$ rig, then $\alpha \in \mathfrak{m}_{\mathcal{O}^{an}_{S,s}}$ &

$$\frac{\hat{\mathcal{O}}^{an}_{S,s}[[u,v]]}{(uv-\alpha)} \cong \hat{\mathcal{O}}^{an}_{C,c}$$

And $\alpha\cdot\mathcal{O}^{an}_{S,s}$ is unique — call it the sing. ideal.

<u>Def</u>: C/S s-stable, $s \in S$. The <u>labeled red'n graph</u> Γ_s of C/S at s is the ~~gpp~~ red'n graph Γ of C_s, with each edge e labelled by the sing. ideal (in $\mathcal{O}_{S,s}^{an}$) of the corresp. singular pt.

<u>Def</u>: We say C/S is <u>aligned</u> at s if for every cycle γ in Γ_s, and every two edges e_1, e_2 in γ with labels l_1, l_2, there exist integers $n_1, n_2 \geq 0$ s.t.

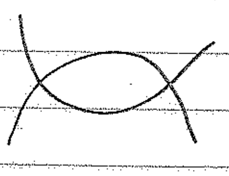
$$l_1^{n_1} = l_2^{n_2} \quad \left(\text{equality of ideals in } \mathcal{O}_{S,s}^{an}\right)$$

We say C/S aligned if aligned at all $s \in S$.

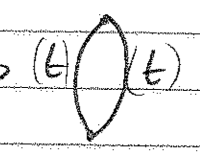
<u>eg</u>: If S is a Dedekind scheme ($\sim$ reg, dim 1) this is automatic since all local rings are DVRs.

- eg $S = \operatorname{Spec} \mathbb{C}[[s,t]] \approx \mathbb{A}^2$.
  interested in $(s=t=0)$.

$$C: y^2 = (x+1)(x+1+t)(x-1)(x-1-t) \subseteq \mathbb{P}(1,1,2)$$



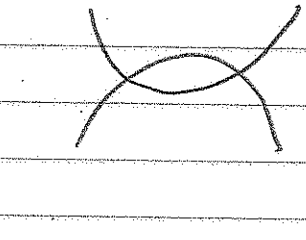
$\rightsquigarrow$



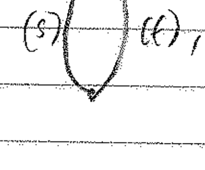
$(t)^1 = (t)^1$, aligned

- eg $S = \operatorname{Spec} \mathbb{C}[[s,t]]$, over $(s=t=0)$

$$C: y^2 = (x+1)(x+1+s)(x-1)(x-1-t)$$



$\rightsquigarrow$



$(s)^n \neq (t)^m$ if $n, m \neq 0$

$\rightsquigarrow$ <u>NOT</u> aligned.

- <u>eg</u>: compact type $\Rightarrow$ aligned

8

S separated &

<u>Thm</u>: Let ~~[illegible]~~ C/S s-stable, with C regular, & smooth over $U \subseteq S$ d. open. Then the jacobian $J = \mathrm{Jac}_{C_U/U}$ admits a Néron model over S iff C/S is aligned, & the NM is of f. type if exists.

('NM iff aligned').

<u>Thm</u>: Let S separated, ~~[illegible]~~ & normal, & C/S semistable, & smooth over some $U \subseteq S$. ~~Then~~ ~~$J = \mathrm{Jac}_{C_U/U}$~~ ~~$\exists V \subseteq S$ open s.t.~~ ~~Let $V \subseteq S$ open be set of all aligned points~~. Then $\exists V \subseteq S$ open s.t. $\mathrm{codim}_S (S \setminus V) \geq 2$, ~~[illegible]~~ & s.t.

$\mathrm{Jac}_{C_U/U}$ admits a (fin. type) NM over V.

('NM exists outside codim 2').

<u>Thm</u>: Let C/S semistable, & $S' \to S$ proper surjective. Then C/S aligned $\iff$ $C \times_S S' \to S'$ aligned.

('blowing up base doesn't help').

~~[illegible]~~

If time allows (!).

- $M_{g,n}$ = moduli of curve of genus g w. n marked pts
- $\overline{M}_{g,n}$ = DM-cpct

$C := \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ univ. curve, smooth over ~~the~~ $M_{g,n}$.

- $J_{g,n} := \cancel{\mathrm{Jac}(\overline{M}_{g,n+1}/\overline{M}_{g,n})}\ \mathrm{Jac}(C_{M_{g,n}}/M_{g,n})$

~~Does $J_{g,n}$~~ In gen, $J_{g,n}$ has no ~~NM~~ NM over ~~the~~ $\overline{M}_{g,n}$.

Def: Given $f : T \to \overline{M}_{g,n}$, s.t. $f^{-1}M_{g,n}$ dense in T & T regular, say T is 'NM admitting' if $f^{*}J_{g,n}$ has a NM$/T$.

Thm The 2-cat of NM admitting morphisms has a terminal object, denoted $\widetilde{M}_{g,n} \to M_{g,n}$.