D. Holmes

Families of elliptic curves over bases of various dimensions:

General setup: · S reg. sep scheme, $U \subseteq S$ dense open, $E/_U$ elliptic.

eg: $S = \operatorname{Spec} k$ ~~alg~~, $U = S$

$S = \mathbb{A}^1_k$, $U = \mathbb{A}^1 \setminus$ origin — $S = \operatorname{Spec} \mathbb{Z}$ or modular curve

$S = \mathbb{A}^2_k$, $U = \mathbb{A}^2 \setminus$ coord-axes. — $S =$ modular curve$/_{\mathbb{Z}}$.

· If $\dim S = 0$ (say a pt), then all is nice - $U = S$, & we have sm. proj. alg. gp.

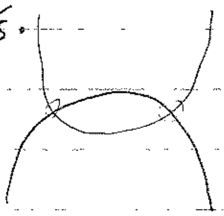
· If $\dim S = 1$; eg. $S =$ small nbd of origin in $\mathbb{A}^1_{\mathbb{C}}$,

$U = S \setminus \{t = 0\}$

$E: y^2 = ((x-1)^2 - u)((x+1)^2 - u)$ (pick unit section O.

Same eq'n makes sense over S, $\rightsquigarrow \bar{E}/_S$.

Fibre over $u=0$:



Let $\bar{E}^{sm}$ = locus where $\bar{E} \to S$ is smooth (ie throw out od).

Then there is a (unique) gp structure on $\bar{E}^{sm}$ extending that on E.

More generally:

Thm: {Neron 5?} let S regsep. scheme of dim 1, $U \subseteq S$ d. open, $A/_U$ ab. scheme. Then $\exists$ a NM (smooth & sep) $\mathcal{A}/_S$ for A.

ie $\mathcal{A}/_S$ comes together with an isom $\mathcal{A}_U \xrightarrow{\sim} A$, such that

NMP: $\forall\, T \to S$ smooth, $\forall\, f: T_U \to A$, $\exists!\ F: T \to \mathcal{A}$ s.t. $F|_{T_U} = f$.

Thm (?) S, U as above, $\bar{E}/_S$ MRM of ell. curve $E/_U$. Then $\bar{E}^{sm} \to S$ is the NM of E.

eg: • Take $T=S$, $T \to S = id$. Then NMP tells you any section in $A(U)$ extends uniquely to a section in $A(S)$.

• Note A automatically a gp scheme.

<u>What if $\dim S > 1$?</u>

eg: $S =$ nbhd of 0 in $\mathbb{A}^2_{\mathbb{C},u,v}$ $U = S \setminus (uv=0)$.

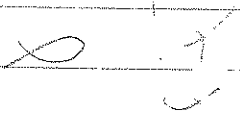
$\bar{E}: y^2 = ((x-1)^2 - u)((x+1)^2 - v))$.   Set $E = \bar{E}|_U$.

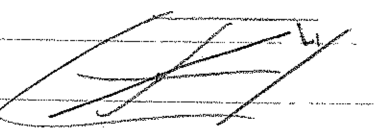
Again, pick a section as unit.

Let $\bar{E}^{sm} \to S$ be smooth locus. Does this have a groupstructure (extending that on E)?

~~Well, if so then we'd get a gp structure~~

First, let's try restricting to lines in S through 0.
Let $\lambda \in \mathbb{C}^*$, & let L_λ be the line in S given by $u = \lambda v$:





Over L_λ, we get ell. curve E_λ outside origin.
• the restriction $\bar{E}_\lambda$ is min. reg. so

$\bar{E}^{sm}_\lambda \to L_\lambda$ is NM of E_λ!

In particular, by restriction we get a gp structure on closed fibre

$$\bar{E}^{sm}_\lambda \Big|_0 .$$

Note $\bar{E}^{sm}_\lambda\big|_0 = \bar{E}^{sm}\big|_0$, so get a gp structure on fibre of $\bar{E}^{sm} \to S$ over closed pt. <u>However</u>, the gp structure depends on λ!

- This is not quite apt, but makes it pretty clear the gp structure on E will not extend to $\bar{E}^{sm}$. Similarly, it is not too hard to show that E does ~~not~~ not have a ~~???~~ NM over S.

Blowing up

The group structure on $\bar{E}^{sm}_0$ we found depended on the 'direction' λ from which we approached the origin, & this obstructs NM.

Natural idea: try blowing up S ~~at~~ at $\underline{o}$, to 'separate out' these directions.

Doesn't work!

Thm [H]: ~~A~~ $U \subseteq S$, E/U as above. Let ~~f~~ $f: \tilde{S} \to S$ proper surj, ~~& let $\tilde{E} = f^* E$~~ let $\tilde{U} = f^{-1}U$, let $\tilde{E} = f^* E$, an E.C. over $\tilde{U}$. Then $\tilde{E}$ does not admit a NM over $\tilde{S}$.

~~Idea~~ Classification

We actually do something more general: we classify exactly when jacobians of ~~...~~ ~~semistable~~ nodal curves admit NMs.

Labelled graphs.

S reg. sep, $C \to S$ ~~semistable~~ nodal curve:

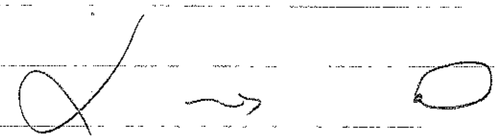
- geom fib red + conn
- nonsm. pts in fibres ~~have a~~ look (étale) locally like $\frac{k[x,y]}{(xy)}$.

eg node
eg not cusp

Reduction graph

k sep. cl. field, $C/_{k}$ nodal. Draw a graph with

- 1 vertex for each irred comp of C
- for each sing. pt of C, an edge between the irred. comps containing it (either 1 or 2).

eg: 

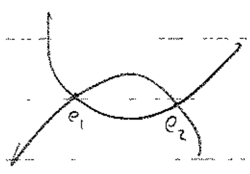
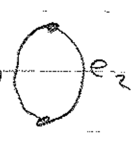
Labelling: Let S scheme, C/S nodal, $\bar{s} \in S$ geom. pt

$\Gamma_{\bar{s}}$ = graph of $C_{\bar{s}}$. Let $e \in \Gamma_{\bar{s}}$ an edge, $\rightsquigarrow$ singular pt on fibre $C_{\bar{s}}$.

Lemma: $\exists$ an element $\alpha \in \mathcal{O}^{et}_{S,\bar{s}}$, unique up to units, such that $\exists$ an $\simeq$ of complete loc. rings

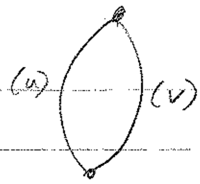
$$\frac{\widehat{\mathcal{O}}^{et}_{S,\bar{s}}[[u,v]]}{(uv-\alpha)} \simeq \widehat{\mathcal{O}}^{et}_{C,e}.$$

~~Pf~~ Eg: $y^2 = ((x-1)^2-u)((x+1)^2-v)$. $\bar{s} = (u=v=0)$

$C_{\bar{s}}$: $y^2 = (x-1)^2(x+1)^2$:  $\Gamma_{\bar{s}}$ = 

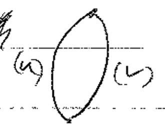
label at e_1 is $\underline{u}$, label at $e_2 = \underline{v}$

In this way, can label edges of graph $\Gamma_{\bar s}$ by principal ideals of $\mathcal{O}^{et}_{S,\bar s}$.



Def: We say C/S is aligned at $\bar s \in S$ if for every circuit $\gamma \in \Gamma_{\bar s}$, & for every two labels l_1, l_2 on γ, $\exists$ integer $n_1, n_2 > 0$ s.t. $l_1^{n_1} = l_2^{n_2}$.

Say C/S aligned if aligned at all $\bar s \in S$.

eg: 

Not aligned.

eg if S = ~~spec~~ dedekind, then $\mathcal{O}^{et}_{S,\bar s}$ a DVR, so if curve gen. sm. then all labels powers of unif, so always aligned.

eg if graph a tree (no loops) then aligned. ('cpt type').

Thm [H]: S reg sep, $U \subseteq S$ d. open, C/S nodal, s.t. $C_U \to U$ smooth. $J = \mathrm{Jac}_{C_U}$.

- If C regular & $C \to S$ aligned, then J has NM/S.
- If $C \to S$ not aligned, then J has no NM/S
- If $f : \tilde S \to S$ prop surj & $\tilde S$ reg, then C/S aligned iff $f^*C/\tilde S$ aligned.

Cor: • Our prev. eg has no NM, even after blowing up / alterings.

• NMs exist over dedekind schemes ~~(known due to Néron)~~ } known anyway.

• NMs exist for cpt type ~~(Heavy & other pf!)~~

Other apps:

{w Owen & Robin}

• Combine w analytic stuff + ~~other~~ electrical network analysis to prove a ~~(special)~~ (case of) conj. of Hain on effectivity of ht jump in Hodge theory;

• [W. O&R]: New pf of thm of S&T on heights in families (state if time).

• New cpt $\widetilde{M}_{g,n}$ of moduli of ~~stable~~ sm. proper curves of genus g w. n marked pts

• $\exists$ canonical map $\widetilde{M}_{g,n} \to \overline{M}_{g,n}$;

• The jacobian of the univ. curve over $\overline{M}_{g,n}^{\,ns}$ has NM over $\widetilde{M}_{g,n}$, & latter is universal wrt this property;

• some other partial cpct (eg. Mamo ?, Grushevsky - Zakharov) fit inside $\widetilde{M}_{g,n}$ ...

• UBC