Néron models: what and why?

Take an elliptic curve, say $E : y^2 = x^3 + x^2 + 7 \in \mathbb{P}^2$.

There is a group law on the rational points: say 3 pts sum to zero iff they are collinear (need to choose a flex pt as origin).

The equation for $E$ makes sense (in general, we can scale coefficients), and so we can consider the surface

$$E_{\mathbb{Z}} = \text{Spec} \left( \mathbb{Z}[x_0, x_1, y] / (y^2 - x_3^2 - x_2^2 - 7) \right)$$

(take closure in $\mathbb{P}^2$).

The group law on $E_{\mathbb{Q}}$ can be written in terms of polynomial equations. If $P_i = (x_i, y_i)$, and say $P_1 + P_2 = P_3$, then generally we have

$$x_3 = \lambda^2 + 1 - x_1 - x_2$$
$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}$$
$$y_3 = -\lambda x_3 - \left( y_1 x_2 - y_2 x_1 \right) \frac{x_2 - x_1}{x_2 - x_1}$$

We can again view these as living in $\mathbb{P}(x_0, x_1, y)$.

**Question:** Does this give us a group law on $E_{\mathbb{Z}_p}$?

Given all these equations over $\mathbb{Z}_p$, we can reduce mod $p$ to get a curve $E_{\mathbb{F}_p}$ and equations for addition.

**Simpler Question:** Do these equations give us a group law on $E_{\mathbb{F}_p}$?
Answer: It depends!

- If $E_p$ is non-singular, then yes.
- If $E_p$ is singular, then the surface $E_p$ may not be regular; in this case, blow up to resolve singularities on $E_p$.
  
  Do (minimal) regular model $\overline{E_p}$.

[See end of notes for comments on smoothness, regularity...]

Then we get a gp law on the smooth pts of $E_{\overline{p}}$.

Note: Smooth pts of $E_{\overline{p}}$ lift to pts of $E_p$ AND vice versa. If we have a regular surface.

Note: If $E_p$ is minimal regular.

- The group law on $E_p$ can be described intrinsically by isomorphism classes of invertible sheaves on $E_p$.
- Everything we said above about reducing equations for the group law mod $p$ can be phrased intrinsically by isomorphism classes of invertible sheaves on $E_{\overline{p}}$ (and the same holds for $E_p$).

However, the fact that equations exist is important in proving representability of the $p$-Picard functor (see later talks...)

In practice

- Why is this useful? Because we can deal with higher genus curves (many other answers!)
Example: $E_7: y^2 = x^3 + x^2 + p$, $E_7': y^2 = x^3 + x^2$. There is no regular point.

We can construct a morphism from $\mathbb{P}^1_{\mathbb{F}_p}$ to $E_{F_p}$ which is birational as follows:

$$\mathbb{P}^1_{\mathbb{F}_p} \rightarrow E_{\mathbb{F}_p}$$

$$t \mapsto (t^2 - 1, t^3 - t) \quad \text{[check, check]}$$

The two pts $t = 1$ & $t = -1$ on $\mathbb{P}^1$ are both sent to the same pt, elsewhere the map has a regular inverse.

Fact: $E_8: y^2 = x^3 + x^2 + p$ is regular. (Yet it is a surface)

They're ruled for addition give $t_1 \circ t_2$:

$$t_1 \mapsto (t_1^2, y)$$. \[check - may have sign or something missing.\]

Magic: Once everything is correctly defined, we can consider arbitrary genus; e.g. $y^2 = x^9 + p$. (g = 4).

Say: the generic smooth fibre contains intrawebles.

Easy calculation: The special fibre of the Neron model of the Jacobian is $E_{\mathbb{F}_p}$ with $\mathbb{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (unipotent).
In general, you get an extension of the Jacobian of the normalization of the special fibre (see also talk) by a group like this.