Definition 8.2.4 Let X be an irreducible curve, $P \in X$ and $f \in K(X)^\times$. Then choose U affine open containing P, and $g, h \in \mathcal{O}_X(U)$ such that $f = g/h$ (Proposition 6.5.3) such that g and h have no zeros on $U - \{P\}$ and define $v_P(f) = v_P(g) - v_P(h)$. We call $v_P(f)$ the *order of vanishing* or *valuation* of f at P.

Remark 8.2.5 Definition 8.2.4 is compatible with Definition 7.5.2. But note once more that in the present section we are not (yet) assuming that X is smooth. If X is not smooth at P, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 > 1$ and $\mathcal{O}_{X,P}$ is *not* a discrete valuation ring.

Definition 8.2.6 Let X be a curve. A *divisor* on X is a $\mathbb{Z}$-valued function D on X such that for at most finitely many P in X, $D(P) \neq 0$. In other words, it is a function $D\colon X \to \mathbb{Z}$ with finite support. The $\mathbb{Z}$-module of divisors is $\mathbb{Z}^{(X)}$, the free $\mathbb{Z}$-module with basis X. Often a divisor D is written as a formal finite sum $D = \sum_{P\in X} D(P)\cdot P$. The degree of a divisor D is defined as $\deg(D) = \sum_P D(P)$.

Example 8.2.7 A typical element of $\mathbb{Z}^{(X)}$ looks something like $2P + 3Q - R$ for some $P, Q, R \in X$. The degree of this divisor is 4.

Lemma 8.2.8 *Let X be an irreducible curve, and f in $K(X)^\times$. Then the set of P in X with $v_P(f) \neq 0$ is finite.*

Proof Recall that our standing assumption is that curves are quasi-projective. Hence X can be covered by finitely many nonempty open affines U_i, such that for each of them, $f|_{U_i} = g_i/h_i$ with g_i and h_i in $\mathcal{O}_X(U_i)$, both non-zero. For each i, U_i is irreducible and affine and of dimension one, hence $Z(g_i)$ and $Z(h_i)$ are zero-dimensional affine varieties, hence finite. □

Definition 8.2.9 Let $f \in K(X)^\times$. Then we define the *divisor of f* as $\mathrm{div}(f) = \sum_{P\in X} v_P(f)P$.

Theorem 8.2.10 *Let X be an irreducible curve. The map $K(X)^\times \to \mathbb{Z}^{(X)}$, $f \mapsto \mathrm{div}(f)$, is a group morphism.*

Proof This is a direct consequence of Proposition 8.2.3 iii. □

Definition 8.2.11 Let X be an irreducible curve, and D and D' divisors on X. Then we say that $D \leq D'$ if for all $P \in X$, $D(P) \leq D'(P)$. This relation "$\leq$" is a partial ordering.

Example 8.2.12 Let P, Q and R be disctinct points on X. Then $P - 3Q + R \leq 2P - 2Q + R$. Note however that $P + Q \not\leq 2Q$ and that $2Q \not\leq P + Q$, so the partial ordering is not a total ordering.

From now on in this chapter we work with smooth curves.

Definition 8.2.13 For X an irreducible smooth curve, D a divisor on X, and $U \subset X$ open and non-empty, we define

$$\mathcal{L}(U, \mathcal{O}_X(D)) := \{f \in K(X)^\times : \mathrm{div}(f|_U) + D|_U \geq 0\} \cup \{0\}.$$

We will often abbreviate $\mathcal{L}(U, \mathcal{O}_X(D))$ to $\mathcal{L}(U, D)$ and $\mathcal{L}(U, \mathcal{O}_X(0))$ to $\mathcal{L}(U, \mathcal{O}_X)$.

Example 8.2.14 Let X be an irreducible smooth curve, $U \subset X$ open and non-empty, and P in X. If P is not in U then $\mathcal{L}(U, P)$ is the set of rational functions f with no pole in U. If P is in U, then $\mathcal{L}(U, P)$ is the set of rational functions f with a pole of order at most 1 at P and no other poles in U.

We will state the following result without proof.

Proposition 8.2.15 *Let X be an irreducible smooth curve.*

i. If X is projective then $\mathcal{L}(X, D)$ is a k-vector space of finite dimension.

ii. If $U \subset X$ is open and non-empty, then $\mathcal{L}(U, \mathcal{O}_X) = \mathcal{O}_X(U)$.

The reader with some background in commutative algebra (especially, localization) may want to prove item (ii) in this result as follows. Let $P \in U$. As X is smooth at P we have that $\mathcal{O}_{X,P}$ is a discrete valuation ring and in particular we have $\mathcal{O}_{X,P} = \{f \in K(X)^\times : v_P(f) \geq 0\} \cup \{0\}$. It follows that $\mathcal{L}(U, \mathcal{O}_X)$ is equal to the intersection of all $\mathcal{O}_{X,P}$ for P running through U. Now a general result in commutative algebra (try to prove this yourself!) states that if R is a domain, then $R = \cap_{\mathfrak{m}} R_{\mathfrak{m}}$, where the intersection is taken inside the fraction field of R and runs over all maximal ideals $\mathfrak{m}$ of R. Here $R_{\mathfrak{m}}$ denotes the localization of R at $\mathfrak{m}$. We obtain (ii) by applying this result to the domain $\mathcal{O}_X(U)$, and by noting that $\mathcal{O}_X(U)_{\mathfrak{m}_P}$ is identified with $\mathcal{O}_{X,P}$ for all $P \in U$.

Example 8.2.16 One may be tempted to believe that even if X is not necessarily smooth, one has that $\{0\} \cup \{f \in K(X)^\times : \mathrm{div}(f) \geq 0\} = \mathcal{O}_X(X)$. This is not true as the following example shows. Let A be the sub-k-algebra $k[t^2, t^3]$ of $k[t]$. It is finitely generated and it is an integral domain. Let X be the affine variety such that $\mathcal{O}_X(X) = A$; it is irreducible. Then $\{0\} \cup \{f \in K(X)^\times : \mathrm{div}(f) \geq 0\} = k[t]$, which is strictly larger than A. Note that X is the curve $Z(y^2 - x^3)$ in $\mathbb{A}^2$ which has a "cusp" at the origin (the morphism $k[x, y] \to A$, $x \mapsto t^2$, $y \mapsto t^3$ is surjective and has kernel $(y^2 - x^3)$).

Corollary 8.2.17 *Let X be a smooth irreducible projective curve. Then $\mathcal{O}_X(X) = \mathcal{L}(X, 0) = k$.*

Proof Proposition 8.2.15 gives that $\mathcal{O}_X(X) = \mathcal{L}(X, \mathcal{O}_X)$, and that this is a finite dimensional k-vector space. It is a sub-k-algebra of $K(X)$, hence an integral domain. Hence it is a field (indeed, for f nonzero in $\mathcal{O}(X)$, multiplication by f on $\mathcal{O}(X)$ is injective, hence surjective, hence there is a g in $\mathcal{O}(X)$ such that $fg = 1$. So, $k \to \mathcal{O}(X)$ is a finite field extension. As k is algebraically closed, $k = \mathcal{O}(X)$. □

8.3 H^0 and H^1

Let X be a smooth irreducible curve. Then there exist nonempty open and affine subsets U_1 and U_2 of X such that $X = U_1 \cup U_2$ (see Exercise 8.5.4).

Definition 8.3.1 Let $H^0(X, \mathcal{O}_X)$ be the kernel of the map

$$\delta \colon \mathcal{L}(U_1, \mathcal{O}_X) \oplus \mathcal{L}(U_2, \mathcal{O}_X) \to \mathcal{L}(U_1 \cap U_2, \mathcal{O}_X),$$

given by $(f_1, f_2) \mapsto f_1|_{U_1 \cap U_2} - f_2|_{U_1 \cap U_2}$. In the same way, we define $H^0(X, D)$ to be the kernel of the map:

$$\delta \colon \mathcal{L}(U_1, D) \oplus \mathcal{L}(U_2, D) \to \mathcal{L}(U_1 \cap U_2, D), \quad (f_1, f_2) \mapsto f_1|_{U_1 \cap U_2} - f_2|_{U_1 \cap U_2}. \tag{8.3.2}$$

Proposition 8.3.3 *We have $H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X)$.*

Proof See Exercise 6.7.8. □

Note that if X is smooth and irreducible, we get $H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X) = \mathcal{L}(X, 0)$. In fact, more generally we have that if X is smooth and irreducible and D is a divisor on X, that $H^0(X, D) = \mathcal{L}(X, D)$. We thus see that we can use the notations H^0 and $\mathcal{L}$ interchangeably. From Proposition 8.2.15 we obtain that if X is moreover projective $H^0(X, D)$ is finite dimensional as a k-vector space. For a different approach we refer to Exercise 8.5.7. For example, if X is smooth, irreducible and projective, we get $H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X) = \mathcal{L}(X, 0) = k$.