Algebraic Geometry–Midterm assignment

- Rules of the game: solutions are due in on 1 November, to be handed in on paper at the beginning of the lecture. Keep a copy of the solutions you hand in. You are strongly encouraged to write your solutions in TEX and hand in a printed version. Include your name, student number, and university on the top of each sheet. If you hand in something handwritten then please try to write very neatly - any parts we cannot easily read will be ignored.
- Your solutions should convince us that you have a good understanding of what has been discussed so far. You may use results from Lectures 1–5, but please give precise references. If you want to use the results of exercises from those lectures, you should include solutions to those exercises. What you hand in should be your own, individual work (you can work together to solve the problems, but please write up your solutions on your own).
- Throughout, k denotes an algebraically closed field, and all varieties we consider are varieties over the field k.

1. When R is a ring (commutative with 1) and $\mathfrak{a} \subseteq R$ is an ideal, we let $J(\mathfrak{a})$ be the intersection of all maximal ideals of R that contain $\mathfrak{a}$.
(a) Show that $J(\mathfrak{a})$ is a radical ideal.
Let X be an affine variety.
(b) Show that the assignments

$$\mathfrak{a} \mapsto Z(\mathfrak{a}) = \{p \in X \,|\, \text{for all}\, f \in \mathfrak{a} : f(p) = 0\}$$

and

$$Z \mapsto I(Z) = \{f \in \mathcal{O}_X(X) \,|\, \text{for all } p \in Z : f(p) = 0\}$$

establish a one-to-one correspondence

$$\{\text{radical ideals of } \mathcal{O}_X(X)\} \rightleftarrows \{\text{closed subsets of } X\}\,.$$

Hint: choose an isomorphism from X to some closed subset of $\mathbb{A}^n$ for some n, and use Theorem 4.1.7.
Now let $\mathfrak{a}, \mathfrak{m}$ be ideals of $\mathcal{O}_X(X)$, with $\mathfrak{m}$ maximal.
(c) Show that $Z(\mathfrak{m}) \subseteq Z(\mathfrak{a}) \iff \mathfrak{m} \supseteq \mathfrak{a}$.
(d) Show that $I(Z(\mathfrak{a})) = J(\mathfrak{a})$.
Let $\varphi\colon X \to Y$ be a morphism of affine varieties, and let $\varphi^*\colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ be the induced map.
Let $Z \subseteq Y$ be a closed subset with ideal $I(Z) \subseteq \mathcal{O}_Y(Y)$.
Let $\varphi^*(I(Z)) \cdot \mathcal{O}_X(X)$ be the ideal of $\mathcal{O}_X(X)$ generated by $\varphi^*(I(Z))$.
(e) Show that the ideal $I(\varphi^{-1}Z)$ of the closed set $\varphi^{-1}Z \subseteq X$ is equal to $J(\varphi^*(I(Z)) \cdot \mathcal{O}_X(X))$.
(f) Give an example of a morphism $\varphi\colon X \to Y$ of affine varieties and a closed set $Z \subseteq Y$ such that $\varphi^*(I(Z)) \cdot \mathcal{O}_X(X)$ is not a radical ideal of $\mathcal{O}_X(X)$.

2. Let Y be a non-empty topological space.
(a) Show that Y contains an irreducible closed subset.
We define the dimension $\dim(Y) \in \{0, 1, \ldots, \infty\}$ of Y to be

$$\sup_{m \in \mathbb{Z}} \{\text{there exist}\, Y_0, \ldots, Y_m \subseteq Y \text{ irreducible and closed such that}\, Y_m \supsetneq Y_{m-1} \supsetneq \cdots \supsetneq Y_0\,\}\,.$$

(b) Give an example of a topological space of infinite dimension.
(c) Give an example of a non-discrete topological space Y such that $\dim(Y) = 0$.
(d) Let Y be a variety of dimension zero. Show that Y is discrete. Hint: reduce to the affine case.