NOTE ON DIVISORS AND THE RIEMANN-ROCH THEOREM

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CONTENTS

1. CARTIER DIVISORS

1.1. **The definition.**

Definition 1.1.1 (Sheaf of meromorphic functions)**.** *For any commutative ring A, we denote $R(A)$ for the non-zero-divisors of A. Let X be a scheme, the sheaf $\mathcal{R}_X$ is defined as: for any open subset $U \subset X$,*

$$\mathcal{R}_X := \{a \in \mathcal{O}_X(U) : \forall x \in U, a_x \in R(\mathcal{O}_{X,x})\}.$$

Moreover, $\mathcal{K}'_X$ is defined to be the presheaf such that

$$\mathcal{K}'_X(U) := \mathcal{R}_X(U)^{-1} \mathcal{O}_X(U),$$

*and $\mathcal{K}_X$ is the sheafification of $\mathcal{K}'_X$. Then we call $\mathcal{K}_X$ the **sheaf of meromorphic functions** on X. We note that if X is integral, then $\mathcal{K}_X$ is just the constant sheaf $K(X)$.*

Lemma 1.1.2. *Let X be a locally noetherian scheme, then for any $x \in X$, $\mathcal{K}'_{X,x} \simeq \operatorname{Frac} \mathcal{O}_{X,x}$, where $\operatorname{Frac} \mathcal{O}_{X,x}$ denote for the totally ring of fraction of $\mathcal{O}_{X,x}$.*

Proof. We have $\mathcal{K}'_{X,x} = \mathcal{R}_{X,x}^{-1} \mathcal{O}_{X,x}$ and $\mathcal{R}_{X,x} \subset R(\mathcal{O}_{X,x})$. It suffices to show $R(\mathcal{O}_{X,x}) \subset \mathcal{R}_{X,x}$ when X is locally noetherian. Let $b_x \in R(\mathcal{O}_{X,x})$, then b_x comes from a $b \in \mathcal{O}_X(U)$ where U is an affine open subset in X. Let I be the annihilator of b, then $I \mathcal{O}_{X,x} = 0$. Since X is locally noetherian, we may assume $\mathcal{O}_X(U)$

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is noetherian, therefore I is finitely generated. This then implies there exists an affine open subset $V \subset U$ such that $I \mathcal{O}_X(V) = 0$. Then $b|_V \in R(\mathcal{O}_X(V))$ and hence $b_x \in \mathcal{R}_{X,x}$. $\square$

Definition 1.1.3 (Cartier divisors)**.** *Let $\mathcal{K}_X^\times$ be the subsheaf of invertible elements of $\mathcal{K}_X$ and $\mathcal{O}_X^\times$ be the subsheaf of invertible elements of $\mathcal{O}_X$. We denote $\mathcal{K}_X^\times / \mathcal{O}_X^\times$ to be the sheafification of the presheaf $U \mapsto \mathcal{K}_X^\times(U)/\mathcal{O}_X^\times(U)$. Then there is a natural morphism $\mathcal{K}_X^\times \to \mathcal{K}_X^\times / \mathcal{O}_X^\times$.*

  (1) *The group of **Cartier divisors** on X is defined to be $\mathrm{Div}(X) := H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$.*
  (2) *The natural morphism above yields a homomorphism*

$$\mathrm{div} : H^0(X, \mathcal{K}_X^\times) \to H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times).$$

  *A Cartier divisor D is called to be a **principal Cartier divisor** if and only if $D \in \mathrm{Im}\,\mathrm{div}$.*
  (3) *We denote the group law on $\mathrm{Div}(X)$ as addition. Then for any $D_1, D_2 \in \mathrm{Div}(X)$, one say D_1 and D_2 are **equivalent**, $D_1 \sim D_2$, if and only if $D_1 - D_2 \in \mathrm{Im}\,\mathrm{div}$.*
  (4) *Let $D \in \mathrm{Div}(X)$. D is said to be **effective** if and only if $D \in \mathrm{Im}(H^0(X, \mathcal{O}_X \cap \mathcal{K}_X^\times) \to H^0(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times))$. We then write $D \geq 0$ and the set of effective Cartier divisors is denoted by $\mathrm{Div}_+(X)$.*
  (5) *$\mathrm{CaCl}(X) := \mathrm{Div}(X)/ \sim$.*

1.1.4. The above definition allows us to represent a Cartier divisors by a system $\{(U_i, f_i)\}$ where $\{U_i\}$ forms a open cover of X and each $f_i \in H^0(U_i, \mathcal{K}_X^\times)$ such that $f_i|_{U_{ij}} \in f_j|_{U_{ij}} \mathcal{O}_X(U_{ij})^\times$, where $U_{ij} = U_i \cap U_j = U_{ji}$.

1.1.5. Suppose now we have two systems $\{(U_i, f_i)\}$ and $\{(V_j, f_j)\}$ which representing a same Cartier divisor D. Then on $U_i \cap V_j$, $f_i = h_{ij} g_j$ for some $h_{ij} \in \mathcal{O}_X(U_i \cap V_j)^\times$. And the converse also holds. Therefore, for convenience, we denote $D = [\{(U_i, f_i)\}]$.

1.1.6. Let $D_1 = [\{(U_i, f_i)\}], D_2 = [\{(V_j, g_j)\}] \in \mathrm{Div}(X)$, then

$$D_1 + D_2 = [\{(U_i \cap V_j, f_i g_j)\}].$$

Additionally, let $D = [\{(U_i, f_i)\}] \in \mathrm{Div}(X)$. Then $D \in \mathrm{Div}_+(X)$ if and only if $f_i \in \mathcal{O}_X(U_i)$. And D is principal if $[\{(U_i, f_i)\}] = [\{(X, f)\}]$.

1.1.7. For any $D \in \mathrm{Div}(X)$, we would like to associate a sheaf to D. Namely, let $D = [\{(U_i, f_i)\}]$, $\mathcal{O}_X(D)$ is the sheaf on X defined by

$$\mathcal{O}_X(D)|_{U_i} = f_i^{-1} \mathcal{O}_X|_{U_i}.$$

Then $D \in \mathrm{Div}_+(X)$ if and only if $\mathcal{O}_X(-D) \subset \mathcal{O}_X$. Moreover, if U is a open subset of X, then $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U)$. We should note that this construction is independent to the choice of the representatives.

1.2. **Cartier divisors and the Picard group.**

Definition 1.2.1 (The Picard group)**.** *Let X be a scheme.*

  (1) *An $\mathcal{O}_X$-module $\mathcal{L}$ on X is **invertible** if for every point $x \in X$, there exists a open neighbourhood U of x in X such that $\mathcal{O}_X|_U \simeq \mathcal{L}|_U$ as $\mathcal{O}_X$-modules. Note that if X is locally noetherian, then this is equivalent to say that $\mathcal{L}$ is coherent and $\mathcal{L}_x$ is free of rank 1 over $\mathcal{O}_{X,x}$.*
  (2) *$\mathrm{Pic}(X)$ denote the set of isomorphism classes of invertible sheaves on X, which is called to be the Picard group of X. Note that the tensor product makes $\mathrm{Pic}(X)$ into a commutative group, whose unit element is the class of $\mathcal{O}_X$.*

Proposition 1.2.2 (Cartier divisors and the Picard group)**.** *Let X be a scheme. Then*

  (1) *The assignment $\rho : D \mapsto \mathcal{O}_X(D)$ is additive, namely,*

$$\rho(D_1 + D_2) = \mathcal{O}_X(D_1) \mathcal{O}_X(D_2) \simeq \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2).$$

  (2) *ρ induces an injective homomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$.*
  (3) *The image of ρ corresponds to the invertible sheaves contained in $\mathcal{K}_X$.*

Proof. (1) Let $D_1 = [\{(U_i, f_i)\}], D_2 = [\{(V_j, g_j)\}] \in \mathrm{Div}(X)$, then $D_1 + D_2 = [\{(U_i \cap V_j, f_i g_j)\}]$. Thus

$$\rho : D_1 + D_2 \mapsto \mathcal{O}_X(D_1 + D_2), \quad \mathcal{O}_X(D_1 + D_2)|_{U_i \cap V_j} = f_i^{-1} g_j^{-1} \mathcal{O}_X|_{U_i \cap V_j}.$$

On the other hand, we may consider

$$D_1 = [\{(U_i \cap V_j, f_i)\}], \quad \mathcal{O}_X(D_1)|_{U_i \cap V_j} = f_i^{-1} \mathcal{O}_X|_{U_i \cap V_j} \quad \text{and}$$
$$D_2 = [\{(U_i \cap V_j, g_j)\}], \quad \mathcal{O}_X(D_2)|_{U_i \cap V_j} = g_j^{-1} \mathcal{O}_X|_{U_i \cap V_j}.$$

Hence the result follows.

(2) For any principal divisor $\mathrm{div}(f)$, it can be represented by $\{(U_i, f|_{U_i})\}$. Then $\mathcal{O}_X(\mathrm{div}(f))|_{U_i} = f^{-1} \mathcal{O}_X|_{U_i}$ yields that $\mathcal{O}_X(\mathrm{div}(f)) \simeq \mathcal{O}_X$ as an $\mathcal{O}_X$-module. Therefore ρ indeed induces a group homomorphism $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$.

Now let $D \in \ker \rho$, then $\mathcal{O}_X(D) \simeq \mathcal{O}_X$ as $\mathcal{O}_X$-modules yields that there exists $f \in H^0(X, \mathcal{K}_X)$ such that $\mathcal{O}_X(D) = f \mathcal{O}_X$. Write $D = [\{(U_i, f_i)\}]$, then $f|_{U_i} = f_i^{-1} \in H^0(U_i, \mathcal{K}_X^\times)$. Therefore $f \in H^0(X, \mathcal{K}_X^\times)$ and $D = \mathrm{div}(f)$ follows immediately.

(3) The construction of $\mathcal{O}_X(D)$ for any $D \in \mathrm{Div}(X)$ yields that $\mathcal{O}_X(D)$ is a locally of free rank one. Let $\mathcal{L} \subset \mathcal{K}_X$ be an invertible subsheaf, and $\{U_i\}$ be an open cover of X such that $\mathcal{L}|_{U_i}$ is free of rank one and is generated by an element $f_i \in \mathcal{K}'_X(U_i)$ for each i. Then $f_i \in \mathcal{K}'_X(U_i)^\times \subset \mathcal{K}_X(U_i)^\times$ because $\mathcal{L}$ is invertible. By letting $D = [\{(U_i, f_i)\}]$, then we obtained the result. $\square$

Corollary 1.2.3. *If X is "nice" enough (e.g. an integral scheme or a reduced noetherian scheme), then $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is an isomorphism. (cf. [Har77, II Proposition 6.15.] and [Liu02, Corollary 7.1.19])*

2. Weil divisors

2.1. **The definition.**

Definition 2.1.1 (Weil divisors)**.** *Let X be a noetherian scheme.*

(1) *A **cycle** on X is an element of the direct sum $\mathbf{Z}^{(X)}$. Thus, for any cycle Z on X, Z can be uniquely written as*

$$Z = \sum_{x \in X} n_x [x].$$

Then we put $\mathrm{mult}_x(Z) := n_x$, the multiplicity of Z at x.

(2) *Let Z be a cycle on X. We say Z is **positive** if $\mathrm{mult}_x(Z) \geq 0$ for all $x \in X$. The **support** of Z, $\mathrm{Supp}\, Z$, is defined to be the union of $\overline{\{x\}}$ where $\mathrm{mult}_x(Z) \neq 0$.*

(3) *A cycle Z on X is **of codimension 1** if every $\overline{\{x\}}$ in $\mathrm{Supp}\, Z$ is of codimension 1. We note that $\overline{\{x\}}$ is of codimension 1 if and only if $\dim \mathcal{O}_{X,x} = 1$. The cycles of codimension 1 then form a subgroup $Z^1(X)$ of the group of cycles on X.*

(4) *Let X be a noetherian integral scheme. Then a cycle of codimension 1 is called a **Weil divisor**.*

(5) *Let X be a normal noetherian (hence integral) scheme, and let $x \in X$ such that $\overline{\{x\}}$ is of codimension 1. Then $\mathcal{O}_{X,x}$ is local of dimension 1 and normal, and it is therefore a discrete valuation ring. We then can define*

$$\mathrm{mult}_x : K(X) \to \mathbf{Z} \cup \{\infty\}$$

by extending the valuation on $\mathcal{O}_{X,x}$. Let $f \in K(X)$ be a non-zero rational function, we define

$$(f) := \sum_{x \in X, \dim \mathcal{O}_{X,x} = 1} \mathrm{mult}_x(f)[x].$$

*Then (f) is a Weil divisor and we call such divisors **principal**.*

(6) *Let X be a normal noetherian scheme. $\mathrm{Cl}(X)$ is defined to be the quotient group of $Z^1(X)$ by the subgroup of principal Weil divisors. Two Weil divisors are then said to be **equivalent** if and only if they are in the same class in $\mathrm{Cl}(X)$, i.e., their difference is principal.*

Example (The ideal class group). Let K be a number field and $X = \mathrm{Spec}\,\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Note that X consists of the prime ideals sitting above the prime numbers in $\mathbf{Z}$ and the zero ideal. Moreover, one can decompose every ideal in $\mathcal{O}_K$ into a product of prime ideals. (cf. [Lan94] or any book of Algebraic Number Theory.) Then for any Weil divisor $D = \sum_{\mathfrak{p} \in \mathrm{Spec}\,\mathcal{O}_K} n_{\mathfrak{p}}[\mathfrak{p}]$, we can associate a fractional ideal $\prod_{\mathfrak{p} \in \mathrm{Spec}\,\mathcal{O}_K} \mathfrak{p}^{n_{\mathfrak{p}}}$ to it. This correspondence then gives (an evidence) that $\mathrm{Cl}(X)$ is isomorphic to the class group of K.

2.2. **Weil divisors and Cartier divisors.**

Definition 2.2.1. *Let A be a commutative ring and M be any A-module.*
  (1) *M is said to be **simple** if $M \neq 0$ and the only sub-A-modules of M are only 0 and itself.*
  (2) *Suppose there exists a chain*
$$0 = M_0 \subset ... \subset M_n = M$$
  of sub-A-modules of M such that each M_{i+1}/M_i is simple, then we put
$$\mathrm{length}_A(M) = n.$$
  Note that this is independent to the choice of the chain.

Lemma 2.2.2. *Let A be a noetherian local ring of dimension 1 and $f, g \in R(A)$. Then A/fA is of finite length and*
$$\mathrm{length}(A/fgA) = \mathrm{length}(A/fA) + \mathrm{length}(A/gA).$$

Proof. We first note that since A is of dimension 1, thus the prime ideals in A is either minimal or maximal. Since f is not a zero-divisor, then f does not belong to any minimal ideal (result from commutative algebra). Note that the prime ideals in A/fA corresponds to the prime ideals in A containing f, therefore we conclude that A/fA is of dimension 0. Hence we have $\sqrt{0} = \mathfrak{m}$, where $\mathfrak{m}$ is the one and only one prime ideal in A/fA. Since A is noetherian, so is A/fA, therefore there exists $n \in \mathbf{Z}_{>0}$ such that $\mathfrak{m}^n = 0$, which implies that A/fA is artinian.

Suppose $A/fA \neq 0$ and contains no simple sub-A-modules. Then we can obtain a descending infinite chain of proper submodules of A/fA, which contradict to A/fA being artinian. Hence let M_1 be a simple submodule of A/fA. If $(A/fA)/M_1 = 0$, then $\mathrm{length}(A/fA) = 1$. If not, then we are facing the case as above but with $(A/fA)/M_1$ rather than A/fA. Inductively, we obtain $M_1 \subset M_2 \subset ... \subset M_n = A/fA$ with M_{i+1}/M_i being simple. The index n is finite because A/fA is noetherian. Hence $\mathrm{length}(A/fA)$ is finite.

Finally, we know that $\mathrm{length}(A/fA)$ and $\mathrm{length}(A/gA)$ are finite, and consider

$$0 \longrightarrow gA/fgA \longrightarrow A/fgA \xrightarrow{\cdot f} A/gA \longrightarrow 0 \,,$$
$$\Big\downarrow \wr$$
$$A/fA$$

then the result follows. $\qquad\square$

2.2.3. Let A be as in the lemma above, then for any $f \in R(A)$, $\mathrm{length}(A/fA)$ is finite integer. Therefore the above lemma allows us to extend the map $f \mapsto \mathrm{length}(A/fA)$ to a group homomorphism $(\mathrm{Frac}\,A)^{\times} \to \mathbf{Z}$. Moreover, if $f \in A^{\times}$, then f is in the kernel of this homomorphism. Therefore we have the group homomorphism

$$\mathrm{mult}_A : (\mathrm{Frac}\,A)^{\times}/A^{\times} \to \mathbf{Z}\,.$$

Definition 2.2.4 (Degrees of Cartier divisors)**.** *Let X be a noetherian scheme and $D \in \mathrm{Div}(X)$. For any $x \in X$ such that $\overline{\{x\}}$ is of codimension 1, the stalk of D at x belongs to $(\mathcal{K}_X^\times / \mathcal{O}_X^\times)_x = (\mathrm{Frac}\,\mathcal{O}_{X,x})^\times / \mathcal{O}_{X,x}^\times$. We then define*

$$\mathrm{mult}_x(D) = \mathrm{mult}_{\mathcal{O}_{X,x}}(D_x), \quad and \quad [D] := \sum_{x \in X, \dim \mathcal{O}_{X,x}=1} \mathrm{mult}_x(D)[x] \in Z^1(X).$$

Proposition 2.2.5. *Let X be a noetherian scheme. Then*

(1) *The assignment $D \mapsto [D]$ establishes a group homomorphism*
$$\mathrm{Div}(X) \to Z^1(X),$$
which sends the effective Cartier divisors to positive cycles.

(2) *Suppose X is now a normal noetherian scheme. Let $f \in H^0(X, \mathcal{K}_X^\times) = K(X)^\times$, then the Weil divisor (f) coincides with the image of the principal Cartier divisor $\mathrm{div}(f)$.*

(3) *Suppose X is again a normal noetherian scheme. Then $\mathrm{Div}(X) \to Z^1(X)$ is injective and induces an injective homomorphism*
$$\mathrm{CaCl}(X) \to \mathrm{Cl}(X).$$

Proof.   (1) It suffices to show that for any $D \in \mathrm{Div}(X)$, there are only finitely many $x \in X$ with $\dim \mathcal{O}_{X,x} = 1$ such that $\mathrm{mult}_x(D) \neq 0$. Let U be an open subset in X such that U is everywhere dens in X and $D|_U = 0$[1]. Then for any $x \in X$ such that $\overline{\{x\}}$ is of codimension 1 and $\mathrm{mult}_x(D) \neq 0$ is a generic point of $X - U$. Therefore in any affine open subset of X, there are only finitely many points x with $\overline{\{x\}}$ of codimension 1 such that $\mathrm{mult}_x(D) \neq 0$. Since X is noetherian, then X can be covered by only finitely many affine opens which are from noetherian rings, hence the result follows.

(2) We have
$$(f) = \sum_x \mathrm{mult}_x(f)[x] \quad and \quad [\mathrm{div}(f)] = \sum_x \mathrm{mult}_x(\mathrm{div}(f))[x],$$
where $\mathrm{mult}_x(\mathrm{div}(f)) = \mathrm{mult}_{\mathcal{O}_{X,x}}(\mathrm{div}(f)_x) = \mathrm{mult}_{\mathcal{O}_{X,x}}(f_x)$. Therefore it is sufficient to show
$$\mathrm{length}(\mathcal{O}_{X,x}/f_x\,\mathcal{O}_{X,x}) = \mathrm{mult}_x(f)$$
for $f \in \mathcal{O}_X(X) \cap K(X)^\times$. Let $t \in \mathcal{O}_{X,x}$ be an uniformiser at x, then $f_x \in (t)^n$ for some $n \in \mathbf{Z}_{\geq 0}$ but $f \notin (t)^{n+1}$. Consider the following chain
$$\mathcal{O}_{X,x} =: (t^0) \supset (t) \supset (t^2) \supset \cdots \supset (t^n) \supset \cdots,$$
where $(t^{i+1})/(t^i)$ is a simple $\mathcal{O}_{X,x}$-module for each i. The image of this chain in $\mathcal{O}_{X,x}/f_x\,\mathcal{O}_{X,x}$ then yields the result.

(3) Let $D = [\{(U_i, f_i)\}] \in \mathrm{Div}(X)$, and we have $[D|_{U_i}] = (f_i) = \sum_{x \in U_i, \dim \mathcal{O}_{X,x}=1} \mathrm{mult}_x(f_i)[x]$ (thanks to (2)). Then

$$D \geq 0 \text{ on } U_i \Leftrightarrow f_i \in \mathcal{O}_X(U_i) \Leftrightarrow \mathrm{mult}_x(f_i) \geq 0 \text{ for all } x \in U_i \Leftrightarrow [D|_{U_i}] \geq 0, \text{ and}$$

$$D = 0 \text{ on } U_i \Leftrightarrow f_i \in \mathcal{O}_X(U_i)^\times \Leftrightarrow \mathrm{mult}_x(f_i) = 0 \text{ for all } x \in U_i \Leftrightarrow [D|_{U_i}] = 0.$$

Hence we are done.

$\square$

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[1]This can always happen. Suppose D is locally given by $f = a/b \in \mathrm{Frac}\,A$ where A is a noetherian ring. Let $\mathfrak{p} \in \mathrm{Spec}\,A$ such that $a_\mathfrak{p} = 0$, then $a \in \mathfrak{p}$. Since $a \in R(A)$ thus $V(a)$ contains no generic points in $\mathrm{Spec}\,A$, hence $D(a)$ is dense in $\mathrm{Spec}\,A$. Apply the same argument b, hence we have $D|_{D(a) \cap D(b)} = 0$ since $a/b \in \mathcal{O}_X(D(a) \cap D(b))^\times$.

Corollary 2.2.6. *Suppose X is a regular integral noetherian scheme (hence normal), then*

$$\mathrm{Div}(X) \to Z^1(X), \quad \mathrm{CaCl}(X) \to \mathrm{Cl}(X)$$

are isomorphisms. (cf. [Liu02, Proposition 7.2.16]*)*

3. The Riemann-Roch theorem

3.1. **The Riemann-Roch theorem.**

3.1.1. Let k be a field. By an **affine variety** over k, we mean an affine scheme associated to a finitely generated algebra over k. Then an **algebraic variety** over k is then a k-scheme which admits a finite affine open cover which are affine varieties over k. Similarly, a **projective variety** over k is a projective scheme over k, and projective varieties are then algebraic varieties. We should furthermore note that the morphisms of varieties are just the morphisms of k-schemes.

3.1.2. By a **curve** over a field k, we mean an algebraic variety whose irreducible components are of dimension 1. In this subsection, we will fix a field k and a curve X over k. We should note that in this situation, the Cartier divisors are isomorphic to the Weil divisors (thanks to Proposition 2.2.5).

Definition 3.1.3. *Let $D \in \mathrm{Div}(X)$, then*

$$\deg_k D := \sum_x \mathrm{mult}_x(D)[k(x) : k]$$

where x runs through all the closed points of X. Then

$$\deg_k : \mathrm{Div}(X) \to \mathbf{Z}$$

is then a group homomorphism.

3.1.4. Let Y be a scheme and $D \in \mathrm{Div}_+(Y)$. Recall that we have defined a sheaf associated to each Cartier divisors in 1.1.7. We then have $\mathcal{O}_Y(-D)$ is a $\mathcal{O}_Y$-module lying in $\mathcal{O}_Y$, and is hence a sheaf of ideals. Therefore we let $(D, \mathcal{O}_D)$ to be the closed subscheme in Y associated to the sheaf of ideals $\mathcal{O}_Y(-D)$.

Lemma 3.1.5. *Let $D \in \mathrm{Div}_+(X)$ be a non-zero Cartier divisor, then*

$$\deg_k D = \dim_k H^0(D, \mathcal{O}_D).$$

Proof. Locally on D is cut out by an element which is not a unit as well as not a zero-divisor. Therefore $D = \mathrm{Spec}\, R/fR$ with R/fR has one and only one prime ideal. Therefore, D consists of closed points in X. Since X is a curve (a variety) over k, and the polynomial rings over k are UFDs, therefore D is finite, and hence affine. Let $A = H^0(D, \mathcal{O}_D)$, then $A = \oplus_{x \in D} \mathcal{O}_D(\{x\})$. Hence $\dim_k A = \sum_{x \in D} \dim_x \mathcal{O}_X(\{x\})$. Exercise 7.1.6(d) (see Section 4) yields that

$$\mathrm{mult}_x(D) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{D,x}) = [k(x) : k]^{-1} \dim_k \mathcal{O}_{D,x}.$$

Hence we obtain the lemma. $\square$

Lemma 3.1.6. *Let Y be a noetherian scheme of dimension 1 and $D \in \mathrm{Div}(Y)$ be a non-zero Cartier divisor on Y. Then there exists two non-zero effective Cartier divisors E, F on Y such that $D = E - F$.*

Proof. Let $D = [\{(U_i, f_i)\}]$ with $f_i = a_i/b_i$. Consider $V(b_i) \subset U_i$. Without loss of generality, we may assume $U_i = \mathrm{Spec}\, A$, where A is noetherian. Since b_i is not a unit and as well as not a zero-divisor, thus $V(b_i)$ has dimension 0 with $\mathcal{O}_{V(b_i)}(V(b_i)) = A/b_i A$ which is also noetherian. Apply [Liu02, Corollary 7.1.3 & Corollary 7.1.5][2], then we have $V(b_i)$ is finite. Since b_i is not a zero-divisor, $V(b_i)$ does not contain generic point of U_i, and $V(b_i)$ is closed in U_i. Let D_i be the Cartier divisor on X defined by the system $\{(X - V(b_i), 1), (U_i, b_i)\}$. Then $F := \sum D_i$ is an effective Cartier divisor, and so is $E := D + F$. Adding a non-zero effective Cartier divisor to F and E if necessary, then we have E, F be non-zero effective Cartier divisors satisfying what we want. $\square$

Theorem 3.1.7 (Riemann)**.** *Now we suppose X is furthermore projective and $D \in \mathrm{Div}(X)$. The Euler-Poincaré characteristic[3] of $\mathcal{O}_X(D)$ is defined to be $\chi_k(\mathcal{O}_X(D)) = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D))$. Similarly we define $\chi_k(\mathcal{O}_X)$ for $\mathcal{O}_X$. Then*

$$\chi_k(\mathcal{O}_X(D)) = \deg_k D + \chi_k(\mathcal{O}_X).$$

Proof. Thanks to Lemma 3.1.6, we can write $D = E - F$, where E, F are non-zero effective Cartier divisors. Then we have an exact sequence of sheaves

$$0 \to \mathcal{O}_X(-F) \to \mathcal{O}_X \to \mathcal{O}_F \to 0.$$

Note that $\mathcal{O}_X(E)$ is flat over $\mathcal{O}_X$, therefore the sequence stay exact after tensoring, that is, we have an exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(E) \, \mathcal{O}_X(E)|_F \to 0.$$

Since F is a finite scheme, thus $\mathcal{O}_X(E)|_F \simeq \mathcal{O}_F$. Therefore

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D)) \longrightarrow H^0(X, \mathcal{O}_X(E)) \longrightarrow H^0(X, \mathcal{O}_F)$$
$$\longrightarrow H^1(X, \mathcal{O}_X(D)) \longrightarrow H^1(X, \mathcal{O}_X(E)) \longrightarrow 0$$

is exact. Then we have

$$0 = \dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(E)) + \dim H^0(X, \mathcal{O}_F) - \dim H^1(X, \mathcal{O}_X(D)) + \dim H^1(X, \mathcal{O}_X(E)).$$

Apply Lemma 3.1.5, we have $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(E)) - \deg F$. Let $D = 0$, then we have $\chi(\mathcal{O}_X(E)) = \chi(\mathcal{O}_X) + \deg E$. Therefore

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg E - \deg F = \chi(\mathcal{O}_X) + \deg D.$$

$\square$

Definition 3.1.8. *Let X be a projective curve over k. The **arithmetic genus** of X is defined to be*

$$g_a(X) := 1 - \chi_k(\mathcal{O}_X).$$

Theorem 3.1.9 (Riemann-Roch)**.** *Let X be a projective curve over k. Thanks to Serre's duality, we have the dualising sheaf ω_X of X over k. Then for any divisor $D \in \mathrm{Div}(X)$, we have*

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \omega_X \otimes \mathcal{O}_X(-D)) = \deg_k D + 1 - g_a(X).$$

3.1.10. Suppose k is an algebraically closed field and X is a smooth irreducible projective curve over k. Then the smoothness of X yields that the tangent space of X at every point is of dimension 1, which allows us to define a discrete valuation on the rational functions $K(X)$. Moreover, the smoothness also yields that $[k(x) : k] = 1$, hence the formula of the degree and the Riemann-Roch theorem then turn out to be the version provided in Silverman's book [Sil94] (as well as in AG1).

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[2]Corollary 7.1.3 gives that minimal prime ideals in a noetherian ring are associated prime ideals, while Corollary 7.1.5 tells us that the associated prime ideals of a finitely generated module over a noetherian ring is finite.

[3]The Euler-Poincaré characteristic can be defined in a more general case:
Let Y be a projective variety over k and $\mathcal{F}$ be a coherent sheaf on Y. Then $\chi_k(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F})$. Note that $H^i(X, \mathcal{F}) = 0$ for all $i > \dim X$, thus the sum is in fact a finite sum.

3.2. **An application to elliptic curves.** A classical application of the Riemann-Roch theorem is to show that every elliptic curve (defined over an algebraically closed field) is defined by a Weierstraß equation. However, one may also want to consider the elliptic curves defined a ring (or even defined over a scheme), and one may come up with the question: Are such "abstract" elliptic curves also associated to Weierstraß equations?

Definition 3.2.1 (Elliptic curves). *Let S be a scheme.*

(1) *By a **curve** C over S, we mean $C \to S$ is a smooth morphism of relative dimension 1 (all the non-empty fibres are of dimension 1), which is separated and of finite presentation, i.e., locally of finite presentation, pullback of quasi-compact sets are quasi-compact, and the pull-back of quasi-compact sets in $C \times_S C$ under $C \to C \times_S C$ is quasi-compact.*

(2) *By an **elliptic curve** E over S, we mean E is a proper smooth curve over S with geometrically connected fibres all of genus 1 and together with a fixed section 0.*

$$
\begin{array}{c}
C \\
0 \left(\right\uparrow \quad \downarrow \pi \\
S
\end{array}
$$

3.2.2. Let S be an arbitrary scheme. We first recall that in 1.1.7, we can consider the effective Cartier divisors as "closed subschemes with the ideal sheaves are locally $\mathcal{O}_X$-modules of rank 1". Now, in order to talk about Cartier divisors on a S-scheme X, we should define the effective Cartier divisors on it with some "nice" structure over S, then using the strategy provided in Lemma 3.1.6 to define the group of Cartier divisors.

Therefore, in [Ka&Ma], a (relative) effective Cartier divisor $D \subset X$ is considered to be a closed subscheme of X which is flat over S and its ideal sheaf is an invertible $\mathcal{O}_X$-module. The flatness then yields the (relative) Cartier divisors being stable under any base changing.

With this notion, locally on S, say $S = \operatorname{Spec} R$, and cover X by open affines $U_i = \operatorname{Spec} R_i$, then $D \cap U_i$ is defined in U_i by cutting out an element $f_i \in R_i$ such that f_i is not a zero-divisor in R_i and R_i/f_iR_i is flat over R. (Compare to 1.1.4.)

Lemma 3.2.3. *Let C be a smooth curve over a scheme S. Then any section $s \in C(S)$ defines an effective Cartier divisor.*

Lemma 3.2.4. *Suppose C is a proper smooth curve over a scheme S, then every effective Cartier divisor is proper over S.*

Definition 3.2.5. *Let C be a smooth curve over a scheme S, and $D \in \operatorname{Div}_+(C)$, which is proper over S. Then locally on S, say $S = \operatorname{Spec} R$, the affine ring of D is a locally free R-module of finite rank. This rank is then constant locally on S. Then we define $\deg D$ to be this rank. (Compare to Lemma 3.1.5.)*

Lemma 3.2.6. *Let C be a smooth curve over a scheme S. Then for any $s \in C(S)$, the associated effective Cartier divisor is proper over S and of degree 1. The converse also holds.*

3.2.7. To apply the Riemann-Roch theorem, we first consider the classical case, where k is an algebraically closed field and C to be a curve defined over k. Consider D to be the trivial divisor, then the Riemann-Roch theorem gives that $\dim_k H^0(X, \omega_X) = g_a(X)$. Secondly, we consider the divisor associated to the canonical sheaf ω_X, denote by K_C. Apply Riemann-Roch to $D = K_C$, then we have $\deg_k K_C = 2 - 2g_a(C)$. Exercise 7.1.13 (see 4) then allows us to conclude $\dim_k H^0(X, D) = \deg D + 1 - g_a(C)$ for divisor D with $\deg D > 2 - 2g_a(C)$.

Now consider the case of an elliptic curve E, which we have $g_a(E) = 1$. Therefore for any effective Cartier divisor D, we have $\dim_k H^0(X, D) = \deg D$. Consider the divisor defined by the section 0, then we have

$$n = 1 \Rightarrow H^0(X, \mathcal{O}_X([0])) \text{ is free on } 1,$$
$$n = 2 \Rightarrow H^0(X, \mathcal{O}_X(2[0])) \text{ is free on } 1, x,$$
$$n = 3 \Rightarrow H^0(X, \mathcal{O}_X(3[0])) \text{ is free on } 1, x, y,$$
$$n = 4 \Rightarrow H^0(X, \mathcal{O}_X(4[0])) \text{ is free on } 1, x, y, x^2,$$
$$n = 5 \Rightarrow H^0(X, \mathcal{O}_X(3[0])) \text{ is free on } 1, x, y, x^2, xy$$
$$n = 6 \Rightarrow H^0(X, \mathcal{O}_X(6[0])) \text{ is free on } 1, x, y, x^2, xy, x^3, \text{ or } 1, x, y, x^2, xy, y^2.$$

Therefore there must be a relation of $1, x, y, x^2, y^2, xy, x^3, y^2$. By a suitable changing of variables, we then have the Weierstraß equation
$$y^2 + a_1 xy + a_3 y = x^3 + a_4 x + a_6.$$

3.2.8. Now we turn our attention to the elliptic curves over an arbitrary scheme. The first glance from Definition 3.2.1 to Lemma 3.2.6 seems to make everything work fine if we consider the Riemann-Roch theorem on the fibres. However, we don't know if the dimensions of the cohomologies vary or not while the fibres varying. That is, we some how need stronger conditions to make what we want really work.

Theorem 3.2.9 (Cohomology and Base Change). *Let $f : X \to Y$ be a projective morphism of noetherian schemes (will smooth proper flat morphisms work?), and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Let $y \in Y$ Then*

(1) *if the natural map*
$$\varphi^i(y) : R^i f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} k(y) \to H^i(X_y, \mathcal{F}_y)$$
is surjective, then it is an isomorphism, and the same is true for all y' in a suitable neighbourhood of y, where X_y is the fibre of y, $\mathcal{F}_y$ is the pull-back of $\mathcal{F}$ via $X_y \to X$, and $k(y)$ is the constant sheaf on $\overline{y} \subset Y$;

(2) *if $\varphi^i(y)$ is surjective, then the following conditions are equivalent:*
   (a) *$\varphi^{i-1}(y)$ is also surjective;*
   (b) *$R^i f_*(\mathcal{F})$ is locally free in a neighbourhood of y.*

(cf. [Har77, III Theorem 12.11])

3.2.10. Let's just assume we are in the case that we can apply this Theorem, as well as the situation that we have the isomorphism
$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^* \otimes \omega_{X/Y})^*$$
where $\omega_{X/Y}$ is the canonical sheaf of $X \to Y$, and $\mathcal{F}$ is a invertible sheaf over X flat over Y.

For elliptic curve $f : E \to S$, we have $H^i(C_s, \mathcal{F}_s) = 0$ for all $i > 1$ and $s \in S$. Therefore Theorem 3.2.9 tells us that
$$R^2 f_*(\mathcal{O}_X(D)) \otimes k(s) = 0,$$
for any relative effective Cartier divisor D. Since $H^1(E_s, \mathcal{O}_X(D)_s) = 0$, we again have $\varphi^1(s)$ being surjective, and thus
$$R^1 f_*(\mathcal{O}_X(D)) \otimes k(s) \simeq H^1(E_s, \mathcal{O}_X(D)_s) = 0.$$
Since $R^1 f_*(\mathcal{O}_X(D))$ is a push-forward of a coherent sheaf flat over S under a proper morphism [Har77, II Caution 5.8.1], it coherent again. By Nakayama's lemma, it is then locally free of rank 0 at s. Therefore $\varphi^0(s)$ is an isomorphism again. If we define the $H^n(X, \mathcal{F}) = 0$ for all $n < 0$ (we need to be careful at this), then we have $\varphi^{-1}(s)$ surjective again, which implies $R^0 f_*(\mathcal{O}_X(D))$ is locally free in a neighbourhood of s. Therefore we can conclude that we can apply the Riemann-Roch Theorem in this manner, and obtain the Weierstraß equation locally as we expect.

4. Selected exercises from Liu's book

Exercise 5.1.12 (d). Let X be a scheme. Show that the invertible sheaves are actually "invertible". That is, for any invertible sheaf $\mathcal{L}$ on X, its inverse is given by $\mathcal{L}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Solution. It is sufficient to show that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^* \to \mathcal{O}_X$ is an isomorphism. This is an isomorphism if and only if for all $x \in X$, the induced homomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{L}_x, \mathcal{O}_{X,x}) \to \mathcal{O}_{X,x}$$

is an isomorphism. The definition of invertible sheaves then give the result immediately. $\qquad\square$

Exercise 7.1.6. Let $(A, \mathfrak{m})$ be a noetherian local ring. Let M be a finitely generated A-module.
   (1) Show that M is simple if and only if $M \simeq A/\mathfrak{m}$.
   (2) Show that M is of finite length if and only if there exists an $r \in \mathbf{Z}_{>0}$ such that $\mathfrak{m}^r M = 0$.
   (3) Suppose M is of finite length. Show that

$$\mathrm{length}_A(M) := \sum_{i \geq 0} \dim_{A/\mathfrak{m}}(\mathfrak{m}^i M/\mathfrak{m}^{i+1}M).$$

   (4) Suppose A is an algebra over a field k. Show that

$$\mathrm{length}_A(M) \dim_k A/\mathfrak{m} = \dim_k M.$$

Solution. (1) M is simple if and only if $M \neq 0$ and the only submodules of M is O and M itself. Then $\mathfrak{m}M = 0$ or M. If $\mathfrak{m}M = M$, then Nakayama's lemma tells us that $M = 0$, which is a contradiction. Therefore $\mathfrak{m}M$ must be 0. Then M turns out to be a $A/\mathfrak{m}$-vector space with finite dimension. However, since M is simple, then $M \simeq A/\mathfrak{m}$. The converse is trivial.
   (2) Without loss of generality, we suppose M is not simple. Consider $M/\mathfrak{m}M$, which is a finite dimensional $A/\mathfrak{m}$-vector space. Then $A/\mathfrak{m}M$ is of finite length, which is equal to its dimension over $A/\mathfrak{m}$. Therefore we can obtain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = \mathfrak{m}M$$

   with M_{i-1}/M_i being simple for all i. Now apply this to all $\mathfrak{m}^j M/\mathfrak{m}^{j+1}M$, And obtain a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = \mathfrak{m}M \supset \cdots \supset \mathfrak{m}^2 M \supset \cdots.$$

   If M is of finite length, then the chain is finite, which implies our procedure ends at some point, and hence $\mathfrak{m}^r M = 0$. The converse holds by the same strategy.
   (3) Suppose M is of finite length, then

$$\underbrace{M \supset \cdots \supset \mathfrak{m}M}_{\dim_{A/\mathfrak{m}} M/\mathfrak{m}M} \supset \cdots \supset \underbrace{\mathfrak{m}^{r-1}M \supset \cdots \supset \mathfrak{m}^r M}_{\dim_{A/\mathfrak{m}} \mathfrak{m}^{r-1}M/\mathfrak{m}^r M = 0}.$$

   Then the formula holds.
   (4) This follows directly from (3) and $A/\mathfrak{m}$ is a field extension of k.

$\qquad\square$

Exercise 7.1.13. Let $\mathcal{L}$ be an invertible sheaf on an integral scheme X. Let $s \in H^0(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X)$ be a non-zero *rational section* of $\mathcal{L}$.
   (1) Let $\{U_i\}$ be an open cover of X such that $\mathcal{L}|_{U_i}$ is free and generated by an element e_i. Show that there exists $f_i \in K(X)^{\times}$ such that $s_{U_i} = e_i f_i$. Show that $\{(U_i, f_i)\}$ defines a Cartier divisor on X, denoted by $\mathrm{div}(s)$. Show that $\mathcal{O}_X(\mathrm{div}(s)) = \mathcal{L}$.
   (2) If $\mathcal{L} = \mathcal{O}_X$, show that $\mathrm{div}(s)$ is the principal divisor associated to s.
   (3) Show that $\mathrm{div}(s) \in \mathrm{Div}_+(X)$ if and only if $s \in H^0(X, \mathcal{L})$.

(4) Let $D \in \text{Div}(X)$. For any open $U \subset X$, show that

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}_X^\times(U) : \text{div}(f) + D|_U \geq 0\} \cup \{0\}.$$

Solution. (1) We have $(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X)|_{U_i} = e_i \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{K}_X = e_i \mathcal{K}_X$. Since X is integral, $\mathcal{K}_X$ is the constant sheaf $K(X)$. Therefore there exists $f_i \in K(X)$ such that $s_{U_i} = e_i f_i$. Since s is non-zero, therefore $f_i \in K(X)^\times$.

To show $\{(U_i, f_i)\}$ defines a Cartier divisor, it suffices to show that $f_i|_{U_{ij}} \in f_j|_{U_{ij}} \mathcal{O}_X(U_{ij})^\times$. We have

$$(e_i f_i)|_{U_{ij}} = s_{U_{ij}} = (e_j f_j)|_{U_{ij}}.$$

Then $f_i = (e_j/e_i) f_j$ on U_{ij}. Since $e_i|_{U_{ij}} \mathcal{O}_X |_{U_{ij}} = e_j|_{U_{ij}} \mathcal{O}_X |_{U_{ij}}$, thus $(e_j/e_i)|_{U_{ij}} \in \mathcal{O}_X(U_{ij})^\times$. Hence $\{(U_i, f_i)\}$ indeed defines a Cartier divisor.

We have

$$\mathcal{O}_X(\text{div}(s))|_{U_i} = f_i^{-1} \mathcal{O}_X |_{U_i} = e_i s_{U_i}^{-1} \mathcal{O}_X |_{U_i}.$$

(2) If $\mathcal{L} = \mathcal{O}_X$, then $s \in H^0(X, \mathcal{K}_X)$, that is $s_{U_i} = f_i$. So nothing really happens here.

(3) Follow the same notation as above, we have

$$\text{div}(s) \in \text{Div}_+(X) \Leftrightarrow f_i \in \mathcal{O}_X(U_i) \text{ for all } i \Leftrightarrow s \in H^0(X, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X) = H^0(X, \mathcal{L}).$$

(4) Without loss of generality, we assume D is not the trivial divisor. Let $\mathcal{L} = \mathcal{O}_X(D)$, then $\mathcal{O}_X(D)|_{U_i} = e_i \mathcal{O}_X |_{U_i}$ with $D = [\{(U_i, e_i^{-1})\}]$. Let U be any open subset of X, then for any $f \in \mathcal{K}_X^\times(U)$

$$\forall i, f \in \mathcal{O}_X(U \cap U_i) = e_i \mathcal{O}_X(U \cap U_i) \Leftrightarrow \forall i, f = e_i g_i \text{ for some } g_i \in \mathcal{O}_X(U \cap U_i)$$

$$\Leftrightarrow \forall i, g_i = f e_i^{-1}$$

$$\Leftrightarrow \text{div}(f) + D|_{U \cap U_i} = [\{U \cap U_i, f e_i^{-1}\}] \text{ with } f e_i^{-1} \in \mathcal{O}_X(U \cap U_i) \forall i$$

$$\Leftrightarrow \text{div}(f) + D|_U \geq 0.$$

$\square$

Exercise 7.2.1. Let A be a normal noetherian ring and $X = \text{Spec } A$. Suppose $\text{Cl}(X) = 0$.

(1) Let $\mathfrak{p}$ be a prime ideal of height 1 in A. Let $f \in \text{Frac } A$ such that $V(\mathfrak{p}) = (f)$ as Weil divisor on X. Show that $\mathfrak{p} = fA$.

(2) Let $f \in A$ be an irreducible element, i.e., if $f = gh$, then g or h is invertible. Let $\mathfrak{p}$ be a prime ideal, minimal among those containing fA. Show that $\mathfrak{p} = fA$.

(3) Show that A is a UFD.

(4) Show that if B is a UFD, then $\text{Cl}(Y) = \text{Pic}(Y) = \text{CaCl}(Y) = 0$, where $Y = \text{Spec } B$.

Solution. (1) We note that $\mathfrak{p}$ is of height 1 means that $\dim A_{\mathfrak{p}} = 1$, which is as same as $V(\mathfrak{p})$ is of codimension 1. Additionally, $\text{mult}_{\mathfrak{q}}(f) = 1$ when $\mathfrak{q} = \mathfrak{p}$, and equals 0 otherwise. Then the image of f under $A \to A_{\mathfrak{p}}$ is a uniformiser of $\mathfrak{p} A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$, i.e., $f A_{\mathfrak{p}} = \mathfrak{p} A_{\mathfrak{p}}$ and is a unit in $A_{\mathfrak{q}}$ under $A \to A_{\mathfrak{q}}$ for other $\mathfrak{q}$. From here, we can conclude that $f \in \mathfrak{p}$. Now consider the composition map $A \to A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p} A_{\mathfrak{p}} = A_{\mathfrak{p}}/f A_{\mathfrak{p}}$. The kernel of this map then forces $fA = \mathfrak{p}$.

(2) We should suppose f is not a unit and not zero. Consider the Weil divisor

$$(f) = \sum_{\mathfrak{q} \in \text{Spec } A, \dim A_{\mathfrak{q}} = 1} \text{mult}_{\mathfrak{q}}(f)[\mathfrak{q}].$$

For any $\mathfrak{q}$ with $\text{mult}_{\mathfrak{q}}(f) \neq 0$, the image if f under $A \to A_{\mathfrak{q}}$ is inside $\mathfrak{q} A_{\mathfrak{q}}$, and hence $f \in \mathfrak{q}$. Therefore $\mathfrak{q} \in V(\mathfrak{p})$, which implies $\mathfrak{p} = \mathfrak{q}$. In $\mathfrak{p} A_{\mathfrak{p}}$, we have $f \in (\pi^r)$ but $f \notin (\pi^{r+1})$ where π is a uniformiser of $\mathfrak{p} A_{\mathfrak{p}}$, which can be chosen to be an element in $\mathfrak{p}$, and some $r \in \mathbf{Z}_{>0}$. Then there exists $b \in A - \mathfrak{p}$, $c \in A$ such that $bf = c\pi^r$. Suppose c is not a unit, then $\pi | b$ since f is irreducible, which leads to a

contradiction. Therefore $r = 1$ and c is a unit. That is, $fA_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. Now apply (1), and we obtain the result.

(3) For any $g \in A$, we have $(g) = \sum_{\mathfrak{p} \in \operatorname{Spec} A, \dim A_\mathfrak{p} = 1} \operatorname{mult}_\mathfrak{p}[\mathfrak{p}]$. Apply (2), we know that every $\mathfrak{p}$ can be generated by one irreducible element, say $f_\mathfrak{p}$. Hence we have

$$\prod_{\operatorname{mult}_\mathfrak{p} \neq 0} f_\mathfrak{p}^{\operatorname{mult}_\mathfrak{p}(g)}$$

has a same divisor with g, which implies that they are different up to multiplication by a unit since $K(X)^\times$ determines the cycles.

(4) Consider the polynomial ring $k[X_1, X_2, ...]$ with infinity variables over a field k. This ring is a UFD but not noetherian, so how can we define the Weil divisor on it?

$\square$

Exercise 7.2.3. Let $A \to B$ be a homomorphism of local rings. $e_{B/A} := \operatorname{length}_B(B/\mathfrak{m}_A B)$ is defined to be the ramification index of B over A. Let $f : X \to Y$ be a morphism of noetherian schemes.

(1) Let $x \in X$ such that $\overline{\{x\}}$ is of codimension 1 and $y = f(x)$. Show that $\dim \mathcal{O}_{Y,y} = 1$ if f is finite surjective, $\dim \mathcal{O}_{Y,y} \leq 1$ if f is flat. Show that $e_{x/y} := e_{\mathcal{O}_{X,x} / \mathcal{O}_{Y,y}}$ is finite when $\mathcal{O}_{Y,y} = 1$.

(2) Let $D \in \operatorname{Div}(Y)$. Show that $\operatorname{mult}_x(f^*(D)) = e_{x/y} \operatorname{mult}_y(D)$ or 0, according to whether y is codimension 1 or 0. Deduce from this and show that

$$[f^*D] = \sum_x e_{x/y} \operatorname{mult}_x(D)[x]$$

where the sum taken over the set of points $x \in X$ such that $\dim \mathcal{O}_{X,x} = 1$ such that $y = f(x)$ is also of $\dim \mathcal{O}_{Y,y} = 1$.

Exercise 7.3.1. Let X be a projective curve over a field k and let k'/k be an extension. Let $p : X_{k'} \to X$ be the projection morphism. Let $D \in \operatorname{Div}(X)$.

(1) Show that D is principal if and only if $H^0(X, \mathcal{K}^\times) \cap L(D) \neq \emptyset$, where $L(D) = H^0(X, \mathcal{O}_X(D))$.

(2) Suppose X is integral. Show that D is principal if and only if p^*D is.

Appendix Appendix A   Some recaps

Here we provide some definitions which may be helpful for the readers who want to recollect some terminologies which were used in this note. We also give a quick review of cohomology of sheaves and the duality theorems.

A.1   A glossary of some definitions.

Definition A.1.1 (Noetherian schemes)**.** *Let X be a scheme.*

(1) *X is said to be **noetherian** if it is a finite union of affine open $\{U_i\}$ such that $\mathcal{O}_X(U_i)$ is a noetherian ring for every i.*

(2) *X is said to be **locally noetherian** if every point admits a noetherian open neighbourhood.*

Definition A.1.2 (Reduced schemes)**.** *Let X be a scheme and $x \in X$. We say X is **reduced at** x if $\mathcal{O}_{X,x}$ is reduced, i.e., it has no non-zero nilpotents. X is said to be **reduced** if it is reduced at everywhere.*

Definition A.1.3 (Integral schemes)**.** *Let X be a scheme and $x \in X$. We say X is **integral at** x if $\mathcal{O}_{X,x}$ is an integral domain. X is then **integral** if it is integral at all its points.*

Definition A.1.4 (Normal schemes)**.** *Let X be a scheme and $x \in X$. We say X is **normal at** x (or x is a normal point) if $\mathcal{O}_{X,x}$ is normal, i.e., it is a integral domain and is integrally closed in $\operatorname{Frac} \mathcal{O}_{X,x}$. X is further to be **integral** if it is irreducible and normal at all its points.*

Definition A.1.5 (Regular schemes). (1) *By a **regular local ring**, we mean a noetherian local ring with the property that the minimal number of the generators of its maximal ideal equals to its Krull dimension.*

(2) *A **regular scheme** is then a scheme such that all its local rings are regular.*

Definition A.1.6 (Proper schemes). *Let $f : X \to Y$ be a morphism of schemes.*

(1) *f is **locally of finite type** if there exist a covering of Y consists of open affine subsets $V_i = \operatorname{Spec} B_i$ such that for for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$ with each A_{ij} is a finitely generated B_i-algebra. f is called to be **of finite type** is in addition each $f^{-1}(V_i)$ can be covered by a finite number of U_{ij}.*

(2) *Recall that f is closed if it maps any closed subset of X onto a closed subset of Y. We say f is **universally closed** if for any based change, $Y' \to Y$, $X \times_Y Y' \to Y'$ stays closed.*

(3) *f is called to be **separated** if the diagonal map $\Delta : X \to X \times_Y X$ is a closed subspace of $X \times_Y X$.*

(4) *f is said to be **proper** if it is of finite type, separated, and universally closed. In this case, we also say X is proper over Y.*

Definition A.1.7 (Coherent sheaves). *Let X be a scheme with structure sheaf $\mathcal{O}_X$.*

(1) *An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be **quasi-coherent** if for all $x \in X$, there is a open neighbourhood U of x such that there exists an exact sequence*

$$\oplus_{\alpha \in A} \mathcal{O}_X |_U \to \oplus_{\beta \in B} \mathcal{O}_X |_U \to \mathcal{F}|_U \to 0,$$

i.e., $\mathcal{F}$ has a local presentation at every $x \in X$.

(2) *$\mathcal{F}$ is furthermore said to be **coherent** if it is locally finitely presented.*

A.2 Cohomology of sheaves.

Definition A.2.1 (Derived functors). *Let X be a scheme. We consider the category of $\mathcal{O}_X$-modules, $\underline{\operatorname{Mod}}(X)$.*

(1) *A sheaf of $\mathcal{O}_X$-module $\mathcal{I}$ is called to be **injective** if the functor $\operatorname{Hom}(\cdot, \mathcal{I})$ is exact.*

(2) *Let $\mathcal{M} \in \underline{\operatorname{Mod}}(X)$. An **injective resolution** $\mathcal{I}^\bullet$ of $\mathcal{M}$ is a complex together with a morphism $\varepsilon : \mathcal{M} \to \mathcal{I}^0$ such that the sequence*

$$0 \to \mathcal{M} \xrightarrow{\varepsilon} \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

is exact.

(3) *Let $\mathcal{F} : \underline{\operatorname{Mod}}(X) \to \underline{\operatorname{Ab}}$ be a covariant left exact functor from the category of $\mathcal{O}_X$-modules to the category of abelian groups. We define the **right derived functors** $R^i \mathcal{F}$ for each $i \in \mathbf{Z}_{\geq 0}$ as*

$$R^i \mathcal{F}(\mathcal{M}) := H^i(\mathcal{F}(\mathcal{I}^\bullet))$$

the i-th cohomology of the complex $\mathcal{F}(I^\bullet)$, where $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{M}$.

Example (Godement resolution). Let X be a scheme, and any $\mathcal{F} \in \underline{\operatorname{Mod}}(X)$. We define the sheaf $\mathbf{G}_\mathcal{F}$ by

$$\mathbf{G}_\mathcal{F}(U) := \prod_{x \in U} \mathcal{F}_x, \quad \forall U \subset X \text{ open.}$$

There then exists a natural morphism $d^{-1} : \mathcal{F} \to \mathbf{G}_\mathcal{F}$. Now we let $\mathcal{G}^0 := \mathbf{G}_\mathcal{F}$ and define $\mathcal{G}^i := \mathbf{G}_{\operatorname{coker} d^{i-1}}$ where $d^{i-1} : \mathcal{G}^{i-2} \to \mathcal{G}^{i-1}$ is the natural morphism occurs by defining it inductively. Now taking the global sections (we denote $\Gamma(X, \cdot)$), we have

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}^0) \to \Gamma(X, \mathcal{G}^1) \to \cdots.$$

Then $R^i \Gamma(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{G}^\bullet))$.

Example (Čech complex). Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be an open covering of the scheme X and $\mathcal{F}$ be a $\mathcal{O}_X$-module. We consider the complex

$$\check{C}^n(\mathcal{U}, \mathcal{F}) := \prod_{i_0, \ldots, i_n \in \Lambda} \mathcal{F}(\cap_{j=o}^n U_{i_j})$$

and

$$d^n : \check{C}^n(\mathcal{U}, \mathcal{F}) \to \check{C}^{n+1}, \quad (s_{i_0, \ldots, i_n}) \mapsto (\sum_{j=0}^{n+1} (-1)^j s_{i_0, \ldots \hat{i_j}, \ldots i_{n+1}})_{i_0 < \ldots < i_{n+1}}.$$

We then obtained a chain complex

$$\check{C}^0(\mathcal{U}, \mathcal{F}) \to \check{C}^1(\mathcal{U}, \mathcal{F}) \to \check{C}^2(\mathcal{U}, \mathcal{F}) \to \cdots .$$

Then the cohomolgy groups of this complex are

$$\check{H}^n(\mathcal{U}, \mathcal{F}) := H^n(\check{C}^\bullet(\mathcal{U}, \mathcal{F}))$$

are called the **Čech cohomology**.

A.2.2. We shall note that the Čech cohomolgy of a sheaf may not coincide with the cohomology defined by the right derived functors. However, it is a fact that if X is a noetherian separated scheme, $\mathcal{U}$ is a finite cover of affine open subsets, and $\mathcal{F}$ is quasi-coherent, then

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \simeq R^n\Gamma(X, \mathcal{F})$$

for all $n \in \mathbf{Z}_{\geq 0}$.

Theorem A.2.3 (Duality for Projective Schemes). *Let X be a projective scheme of dimension n over a field k. Then the dualising sheaf $\omega_{X/k}$ exists, where $\omega_{X/k}$ is a coherent sheaf on X together with a trace morphism $\mathrm{tr} : H^n(X, \omega_{X/k}) \to k$ such that for all coherent sheaf $\mathcal{F}$ on X, the natural pairing*

$$\mathrm{Hom}(\mathcal{F}, \omega_{X/k}) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_{X/k})$$

followed by tr *gives an isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \omega_{X/k}) \xrightarrow{\sim} H^n(X, \mathcal{F})^*.$$

Theorem A.2.4 (Duality for Proper Smooth Schemes). *Let $X \to S$ be a proper smooth morphism of relative dimension r between locally noetherian schemes . Then the r-dualising sheaf $\omega_{X/S}$ exists. (cf. [Har66] and [Mee18])*

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