NOTE ON MODELS OF ALGEBRAIC CURVES

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Contents

In this note, most of the proofs of the statements will not be given, but the author will refer the proofs to Liu's book (or other references). Instead of proofs, the author wishes to provide enough examples to build up the feeling for studying this topic.

1. Models of Algebraic Curves

We first recall some facts from Liu's book:

Theorem 1.1 (Serre's criterion). *Let X be a locally noetherian scheme, which is connect. Then X is normal if and only if X satisfies:*

1. *X is regular at all of its points of codimension ≤ 1.*
2. *For any $x \in X$,* $\operatorname{depth} \mathcal{O}_{X,x} \geq \inf\{2, \dim \mathcal{O}_{X,x}\}$.

(cf. [Liu02, Theorem 8.2.23]*)*

Corollary 1.2. *Let X be a local complete intersection over a regular locally noetherian scheme. Then X is normal if and only if X is normal at the points of codimension* 1*. (cf.* [Liu02, Corollary 8.2.24]*)*

Example 1.3. Let $S = \operatorname{Spec} R$ be an affine Dedekind scheme. Let $F \in R[x,y] - R$, and $B := R[x,y]/(F)$. We would like to know when $\operatorname{Spec} B$ will be normal. Since $\operatorname{Spec} B$ is defined by one equation, it is a locally complete intersection, and hence we can apply the above Corollary and pay out attention to the points of codimension 1 in $\operatorname{Spec} B$. Such points are either closed points in the generic fibre or the generic points in the closed fibres. The situations we are going to face down below are easy for closed points in the generic fibre. Therefore we would like to focus on the generic points in the closed fibres.

Let $s \in S$ be a closed point and $\xi \in X = \operatorname{Spec} B$ be the generic point of the closed fibre X_s. Let t be the uniformiser of $A = \mathcal{O}_{S,s}$. Then ξ corresponds to the prime ideal generated by t and a polynomial $G \in A[x,y]$ such that $\overline{G} \in k(s)[x,y]$ is irreducible. Then one can write

$$F = G^r H_1 + t^s H_2 \quad \text{for } H_1, H_2 \in A[x,y] \text{ and } r, s \in \mathbf{Z}_{>0}$$

with $\overline{H}_1 \notin \overline{G}k(s)[x,y]$ and $\overline{H}_2 \neq 0$. Now we recall from Liu's book:

Corollary 4.2.12. Let $(A, \mathfrak{M})$ be a regular noetherian local ring, and $f \in \mathfrak{M} - \{0\}$. Then A/fA is regular if and only if $f \notin \mathfrak{M}^2$.

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Corollary 4.2.17. Let X be a noetherian scheme. If X is regular, then its connected components are normal.
These two results then yields us to conclude that X is normal at ξ if and only if either $r = 1$; or $s = 1$ and $\overline{H}_2 \notin \overline{G}k(s)[x,y]$.

Now we can begin to discuss the models of algebraic curves.

Definition 1.4. *Let S be a Dedekind scheme of dimension 1 with function field K, and C is a normal connected projective curve over K. Then a* ***model*** *of C over S is a normal fibred surface $\mathcal{C} \to S$ together with an isomorphism $f : \mathcal{C}_\eta \xrightarrow{\sim} C$, where η is the generic point of S.*

Example 1.5. Let q be a square-free integer and C be a projective curve over $\mathbf{Q}$ defined by

$$X^q + Y^q + Z^q = 0.$$

Let $\mathcal{C}$ be the closed subscheme in $\mathbf{P}^2_{\mathbf{Z}}$ defined by the same equation. We claim that $\mathcal{C}$ is a model of C over $\mathbf{Z}$.

To see this, it suffices to show that $\mathcal{C}$ is normal. The Jacobian criterion tells us that $\mathcal{C}$ is regular outside the primes that divide q, then Corollary 4.2.17 tells us it is normal. Therefore we only need to consider the primes $p|q$.

Let $r := q/p$, since q is square-free, thus $p \nmid r$. Then we have

$$\mathcal{C}_p = \operatorname{Proj} \mathbf{F}_p[X,Y,Z]/(X^r + Y^r + Z^r)^p.$$

Then $\mathcal{C}_p$ is irreducible and $(\mathcal{C}_p)_{\text{red}} = V_{\text{proj}}(X^r + Y^r + Z^r)$ in $\mathbf{P}^2_{\mathbf{F}_p}$. Now consider the affine open subscheme $U = \operatorname{Spec} \mathbf{Z}[x,y]/(x^q + y^q + 1)$ of C and the prime ideal corresponds to $\mathcal{C}_p$ is generated by $x^r + y^r + 1$. One has

$$x^q + y^q + 1 = (x^r + y^r + 1)^p - p((x^r + y^r)F(x^r, y^r) + (x^r + y^r + 1)F(x^r + y^r, 1)),$$

with $F(x^r, y^r)$ modulo p is homogeneous and not equal to 0. Then we have $F(x^r, y^r)$ is not divisible by $x^r + y^r + 1$. Applying Example 1.3, $\mathcal{C}_p$ is then normal.

Proposition 1.6. *Suppose S is further affine and C is a smooth projective curve of genus g over K. Then C admits a relatively minimal regular model (resp. a regular model with normal crossing) over S. If moreover $g \geq 1$, then C admits a unique minimal regular model $\mathcal{C}_{\min}$ and a unique minimal regular model with normal crossing. (cf. [Liu02,* Proposition 10.1.8]*)*

Definition 1.7. *Suppose C is a smooth projective curve over K with genus $g \geq 2$. Suppose C admits a minimal regular model $\mathcal{C}_{\min}$ over S (as in the Proposition, S should be affine). Then we call the canonical model $\mathcal{C}_{\text{can}}$ of the minimal surface $\mathcal{C}_{\min}$ the* ***canonical model*** *of C over S.*

Example 1.8. Let C be the projective curve defined by

$$X^4 + Y^4 + Z^4 = 0$$

over $\mathbf{Q}$ (The Fermat curve of degree 4 with genus 3 and Euler-Poincaré characteristc -2). We aim to see what $\mathcal{C}_{\min}$ and $\mathcal{C}_{\text{can}}$ look like in this example.

As same as the strategy shown in the previous Example, we consider $\mathcal{C}$ to be the closed subscheme in $\mathbf{P}^2_{\mathbf{Z}}$ defined by the same equation. Then the Jacobian criterion tells us that $\mathcal{C}$ is regular outside the prime 2. Consider the affine chart $U = D_{\text{proj}}(Z)$, then U is defined by

$$x^4 + y^4 + 1 = 0.$$

In order to make the computation much easier, we consider the change of variables: $v = y + 1$, $u = x - v$. Then our equation becomes

$$u^4 + 2((v^2 - v + 1)^2 + 3v^2u^2 + 2v^3u + 2vu^3) = 0.$$

Applying Example 1.3, $\mathcal{C}$ is then normal and hence a model of C.

Denote $F = u$ and $G = (v^2 - v + 1)^2 + 3v^2u^2 + 2v^3u + 2vu^3$. We note that Corollary 4.2.12 yields that the singular points in U correspond to the zeros of G. Therefore there is one and only one such point q which corresponds to the ideal $(2, u, v^2 - v + 1)$. In particular, $x = 0, y = z = 1$ is not a singular point and hence by symmetry $x = y = 1, z = 0$

is not a singular point. Hence q is the only singular point in U. Now consider the blowing-up $\widetilde{U} \to U$ centre at q. Then [Liu02, Lemma 8.1.4] gives that $\widetilde{U}$ is a union of three affine pieces, given by the following table:

Spec A_1	Spec A_2	Spec A_3
A_1 is a sub-$\mathbf{Z}$-algebra of $K(C)$ $\mathbf{Z}[u, v, u_1, v_1]$ with $2u_1 = u, 2v_1 = v^2 - v + 1$	A_2 is a sub-$\mathbf{Z}$-algebra of $K(C)$ $\mathbf{Z}[u, v, t_1, s_1]$ with $ut_1 = 2, us_1 = v^2 - v + 1$	A_3 is a sub-$\mathbf{Z}$-algebra of $K(C)$ $\mathbf{Z}[u, v, w_1, z_1]$ with $(v^2 - v + 1)w_1 = 2, (v^2 - v + 1)z_1 = u$
After modulo 2 $\begin{cases} v^2 - v + 1 = 0 \\ v_1^2 + 3v^2u_1^2 + v^3u_1 = 0 \end{cases}$	After modulo 2 $\begin{cases} v^2 - v + 1 = 0 \\ t_1^2(s_1^2 + 3v^2 + v^3t_1) = 0 \end{cases}$	After modulo 2 $\begin{cases} (v^2 - v + 1)w_1 = 0 \\ (v^2 - v + 1)z_1^4 + w_1(1 + v^2z_1^2 + w_1v^3z_1) = 0 \end{cases}$
A smooth conic over $\mathbf{F}_4$	An affine line over $\mathbf{F}_4$ of multiplicity 2 A smooth conic over $\mathbf{F}_4$	An affine line over $\mathbf{F}_2$ of multiplicity 4 An affine line over $\mathbf{F}_4$ of multiplicity 2 A smooth conic over $\mathbf{F}_4$

This process gives us an explicit description of the blowing-up $\widetilde{\mathcal{C}}$ centre at q. The fibre of $\widetilde{\mathcal{C}}$ over 2 then has three irreducible $\Gamma_0 \simeq \mathbf{P}^2_{\mathbf{F}_2}$ of multiplicity 4; Γ_1 a smooth conic over $\mathbf{F}_4$; $\Gamma_2 \simeq \mathbf{P}^2_{\mathbf{F}_4}$ of multiplicity 2. Computing the intersection numbers and using the formula given in [Liu02, Proposition 9.1.21], one has

$$\Gamma_0 \cdot \Gamma_2 = 2, \quad \Gamma_0 \cdot \Gamma_1 = 0, \quad \Gamma_1 \cdot \Gamma_2 = 4, \quad \Gamma_0^2 = -1, \quad \Gamma_1^2 = -8, \quad \Gamma_2^2 = -6.$$

Castelnuovo's criterion yields that Γ_0 is an exceptional divisor, and after the contraction along Γ_0, Γ_2' is a singular conic since it is birational to Γ_2 and has rational points over $\mathbf{F}_2$ (by [Liu02, Proposition 9.3.16]). However, Castelnuovo's criterion gives that there is no exceptional component and hence we get the minimal regular model $\mathcal{C}_{\min}$. Moreover, the adjunction formula gives $K_{\mathcal{C}_{\min}/\mathbf{Z}} \cdot \Gamma_2' = 0$, thus the fibre of $\mathcal{C}_{\mathrm{can}}$ over 2 consists of only one irreducible component Γ_1'.

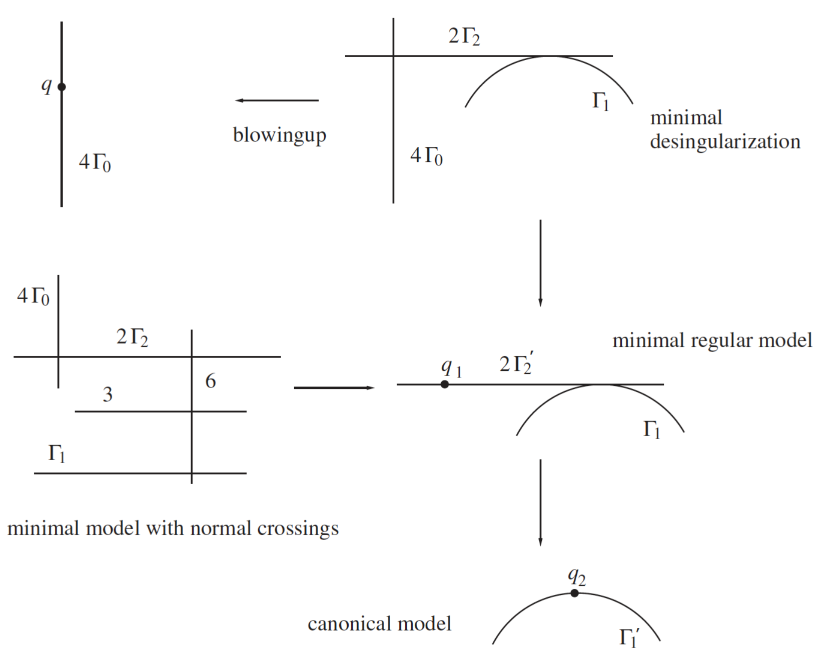

FIGURE 1. The fibres over 2 of models of the curve $X^4 + Y^4 + Z^4$ over $\mathbf{Q}$

Remark. As we have seen in our equation in the above Example, one would like to make the étale base change $\mathbf{Z}[1/3] \hookrightarrow \mathbf{Z}[1/3][X]/(X^2 - X + 1)$. Then for the fibre above the prime number 2, one then has the base change $\mathbf{F}_2 \hookrightarrow \mathbf{F}_4 \simeq \mathbf{F}_2[X]/(X^2 - X + 1)$. If one does so, then we will see two copies of the blowing-up, which are Galois conjugate to each other.

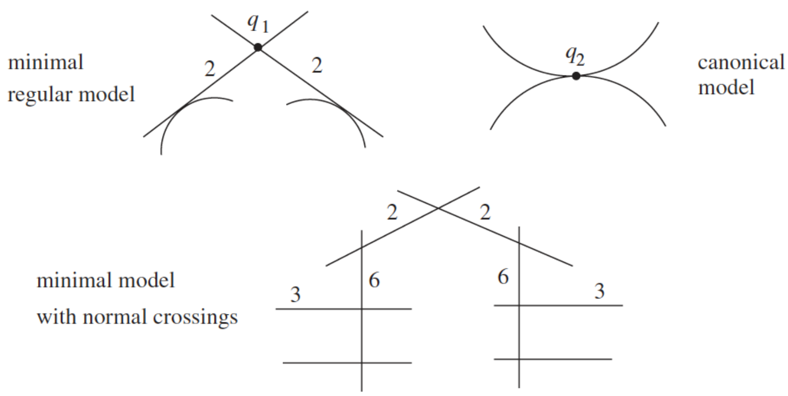

FIGURE 2. The fibres over 2 of models of the curve $X^4 + Y^4 + Z^4$ over $\mathbf{Q}$ after étale base change

2. REDUCTION AND REDUCTION MAP

Definition 2.1. *Let S be a Dedekind scheme of dimension 1 with function field K. Let C be a normal connected projective curve over K.*

(1) *Let $\mathcal{C}$ be a model of C over S and $s \in S$ be a closed point. Then we call the fibre $\mathcal{C}_s$ a* ***reduction*** *of C at s.*
(2) *We say C has a* ***good reduction*** *at s if it admits a smooth model over $\operatorname{Spec} \mathcal{O}_{S,s}$. Otherwise, we say C has a* ***bad reduction*** *at s.*

Example 2.2. Let C be the projective curve over $\mathbf{Q}$ given by $X^3 + Y^3 + p^3 Z^3$ with p being a prime not equal to 3. Then let $\mathcal{C}$ be the projective scheme over $\mathbf{Z}$ by taking the same equation. Then $\mathcal{C}_p$ is a singular curve over $\mathbf{F}_p$. However, by letting $W = pZ$, we also obtain a model of C over $\mathbf{Z}$, which is smooth over $\mathbf{F}_p$. Hence C has a good reduction at p.

This example shows that one cannot directly determine a curve having bad reduction at a point by looking at any model. The following Proposition, however, gives a sufficient and necessary condition to determine whether a curve has good reduction over the base Dedekind scheme or not.

Proposition 2.3. *Let S be a Dedekind scheme of dimension 1 and C be a smooth projective curve over the function field of S with genus $g \geq 1$.*

(1) *C has good reductions at almost all closed points in S, i.e., there exists at most finitely many closed points $s \in S$ such that C has a bad reduction at s.*
(2) *Suppose S is affine. Then C has good reduction over S (i.e., has a good reduction at every closed point in S) if and only if the minimal regular model $\mathcal{C}_{\min}$ of C over S is smooth. Moreover, this implies that $\mathcal{C}_{\min}$ is the unique smooth model of C over S.*
(3) *Let S' be a Dedekind scheme of dimension 1 which is étale over S. Let $s' \in S'$ and s be its image in S .Then $C_{K'}$ has a good reduction at s' if and only if C has a good reduction at s, where K' is the function field of S'.*

Example 2.4. Let C be the projective curve over $\mathbf{Q}$ defined by the equation
$$X^4 + Y^4 + Z^4 = 0.$$

Consider $S = \operatorname{Spec} \mathbf{Z}[1/2]$, then define $\mathcal{C}$ to be the subscheme of $\mathbf{P}^2_{\mathbf{Z}[\frac{1}{2}]}$ defined by the same equation. Then $\mathcal{C}$ is smooth over S, and hence has good reduction over S by applying the above Proposition. Since $S \hookrightarrow \operatorname{Spec} \mathbf{Z}$, thus we may also consider $\mathcal{C}$ as a scheme over $\mathbf{Z}$. This then shows that C has good reductions at primes outside 2. Additionally, as we mentioned in the Example 1.8, $\mathcal{C}_{\min}$ consists of a singular conic, hence is not smooth, hence it has a bad reduction at 2.

Definition 2.5. *Let S be a Dedekind scheme of dimension 1 and K be its function field. Let C be a smooth projective curve over K. Then we say C has* ***potential good reduction*** *at $s \in S$ if there exists a dominate morphism of Dedekind schemes $S' \to S$ and s is the image of some $s' \in S'$ such that $C_{K(S')}$ has a good reduction at s'.*

Example 2.6. Consider C be the projective curve given by $y^2 = x^4 - 1$ over $\mathbf{Q}_2$. We claim that C has potential good reduction over $\mathbf{Z}_2$. To see this, we make the following change of variables:

Set $x = 1 + x_1^{-1}, y = y_1 x_1^{-2} \Rightarrow y_1^2 = 4x_1^3 + 6x_1^2 + 4x_1 + 1$

Set $x_1 = v + \alpha, y_1 = 2z + (\beta v + \gamma)$, where we wish to choose $\alpha, \beta, \gamma \in \overline{\mathbf{Q}}_2$ such that $z^2 + (\beta v + \gamma)z = v^3$.

For such choice of α, β, γ, one consider the system of equations:

$$\begin{cases} 4(1 + 4\alpha + 6\alpha^2 + 4\alpha^3)(6 + 12\alpha) = (4 + 12\alpha + 12\alpha^2)^2 \\ \gamma^2 = 1 + 4\alpha + 6\alpha^2 + 4\alpha^3 \\ \beta^2 = 6 + 12\alpha \end{cases}.$$

The solution exists with

$$|\alpha| = |2|^{-1/4}, \quad |\beta| = |2|^{1/2}, \quad |\gamma - 1| = |2|^{1/4}.$$

Now let $L = \mathbf{Q}_2(\alpha, \beta, \gamma)$ and let $W^0 = \operatorname{Spec} \mathcal{O}_L[v, z]/(z^2 + (\beta v + \gamma)z - v^3)$, then W^0 is smooth over $\mathcal{O}_L$ and its special fibre is an open subscheme of an elliptic curve. Hence we conclude that C_L has good reduction.

Definition 2.7. (1) *A noetherian local ring A is called* ***Henselian*** *if and only if every (module) finite A-algebra is a direct sum of local A-algebras.*

(2) *Let S be the spectrum of a Henselian discrete valuation ring (e.g. complete). Let $\mathcal{X} \to S$ be a surjective proper morphism with generic fibre X. Let X^0 be the set of closed points of X and s the closed point in S. We define the* ***reduction map*** *as*

$$\mathrm{red} : X^0 \to \mathcal{X}, \quad x \mapsto \bullet \in \overline{\{x\}} \cap \mathcal{X}_s\,.$$

We note that the closure is taken in $\mathcal{X}$ and the reduction map depends on the choice of $\mathcal{X}$.

Remark. Here we should note that the map is well-defined. To see this, we first note that $\overline{\{x\}}$ is closed and $\mathcal{X} \to S$ is proper, thus the image of $\overline{\{x\}}$ is closed, and hence is S. Hence $\overline{\{x\}}$ does intersect with X_s. Now let $S = \operatorname{Spec} R$ and we have the following diagram locally:

$$\begin{tikzcd}
 & & B \arrow[rr, hook] & & B_K \\
 & A \arrow[ur, two heads] \arrow[rr] & & A_K \arrow[ur, two heads] & \\
I \arrow[ur, hook] \arrow[rr] & & I_K \arrow[ur, hook] & & \\
 & R \arrow[uu] \arrow[rr, hook] & & K \arrow[uu] &
\end{tikzcd},$$

which is Cartesian everywhere. Note that B_K is a K vector space, and hence is flat over K. Since x is a closed point in X, thus I_K is a maximal ideal, and hence B_K is a field. Since $B \hookrightarrow B_K$, therefore it is reduced. Therefore we can conclude that $\overline{\{x\}}$ is flat over S by [Liu02, Proposition 4.3.9]. Then the dimension of each fibre will be equal, hence $\overline{\{x\}} \cap \mathcal{X}_s$ is finite, and hence $\overline{\{x\}} \to S$ is quasi-finite. Since $\overline{\{x\}} \to S$ is moreover proper, thus $\overline{\{x\}}$ is finite over S. Now since R is Henselian, therefore A in our diagram is then a finite direct sum of local R-algebras. Since $\overline{\{x\}}$ is irreducible, therefore there should be only one piece, and implies $\overline{\{x\}}$ is a local scheme.

Lemma 2.8. *Let $\mathcal{O}_K$ be a Henselian discrete valuation ring, x be a closed point of the generic fibre of $\mathcal{X} = \operatorname{Proj} \mathcal{O}_K[T_0, ..., T_n]$ with homogeneous coordinates $(x_0, ..., x_n) \in \mathcal{O}_{\overline{K}}^{n+1}$ with at least one $x_i \in \mathcal{O}_{\overline{K}}^{\times}$. Then $\operatorname{red}_{\mathcal{X}}(x) = (\widetilde{x_0}, ..., \widetilde{x_n})$.*

Proof. [1] Without lose of generality, one may assume $i = 0$ and $x_0 = 1$. Then on $x \in D_{\text{proj}}(T_0)$, and is defined by the ideal $(T_i/T_0 - x_i)_{i=1}^n$ in $\overline{K}[T_1/T_0, ..., T_n/T_0]$. Then $\overline{x}$ is defined by the same ideal in $\mathcal{O}_{\overline{K}}[T_1/T_0, ..., T_n/T_0]$. Since $\mathcal{O}_{\overline{K}}[T_1/T_0, ..., T_n/T_0]/(T_i/T_0 - x_i)_{i=1}^n \simeq \mathcal{O}_{\overline{K}}$, therefore $(T_i/T_0 - x_i)_{i=1}^n$ is a prime ideal. Therefore we have $\operatorname{red}_{\mathcal{X}}(x) = (\widetilde{x_1}, ..., \widetilde{x_n})$ in $\mathbf{A}^n_{\mathcal{O}_{\overline{K}}}$. Now we need to show that closure of x in $D_{\text{proj}}(T_0)$ coincides with the closure of x in $\mathcal{X}$. However, $D_{\text{proj}}(T_0) \hookrightarrow \mathcal{X}$ is an open immersion, therefore the closure of x in $D_{\text{proj}}(T_0)$ coincides with the closure of x in $\mathcal{X}$. $\square$

Corollary 2.9. *Let C be a normal projective curve over the field of fraction of a Henselian discrete valuation ring $\mathcal{O}_K$, and $\mathcal{C} \subset \operatorname{Proj} \mathcal{O}_K[T_0, ..., T_n]$ be a model of C. Let $x = (x_0, ..., x_n)$ be a closed point of C with homogeneous coordinated $x_i \in \mathcal{O}_{\overline{K}}$ of which at least one belongs to $\mathcal{O}_{\overline{K}}^{\times}$. Then $\operatorname{red}(x) = (\widetilde{x_0}, ..., \widetilde{x_n})$.*

3. Graphs

Definition 3.1. *Let $C > 0$ be a vertical divisor contained in a closed fibre X_s of a regular fibred surface $X \to S$. Let $\Gamma_1, ..., \Gamma_n$ be the irreducible components of C. The* ***dual graph*** *G associated to C is given by:*

- *The vertices of G are the irreducible components $\Gamma_1, ..., \Gamma_n$.*
- *Between two different vertices v_i, v_j (associated to Γ_i, Γ_j respectively), there are $\Gamma_i \cdot \Gamma_j$ edges, where $\Gamma_i \cdot \Gamma_j$ is the intersection number of Γ_i and Γ_j.*

Example 3.2. The following figure shows that the two curves on the left admit the same dual graph (on the right).

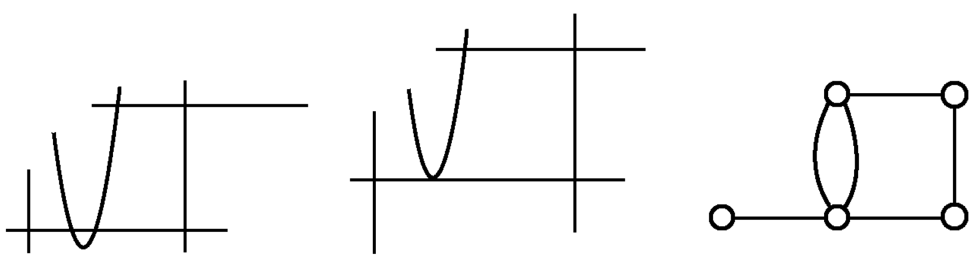

FIGURE 3. Two curves having the same dual graph

Lemma 3.3. *Let $(A, \mathfrak{M})$ be a regular local ring of dimension 2 and let $f_1, ..., f_n \in \mathfrak{M}$ be pairwise relatively prime irreducible elements. Then the natural morphism*

$$\varphi : A/(f_1 \cdots f_n) \to \oplus_{i=1}^n A/(f_i)$$

is injective and its cokernel is of length $\sum_{i<j} \text{length}_A A/(f_i, f_j)$.

Proposition 3.4. *Let C be as in the definition and $C \leq X_s$. Let G be its dual graph. Then*

(1) *C is connected if and only if G is connected.*

[1]This proof is not as same as in Liu's book (cf. [Liu02, Lemma 10.1.32]) since the author could not understand Liu's proof. So any explanation to the author to Liu's proof is welcomed.

(2) *The Betti number of G is defined by*

$$\beta(G) := \#\{\textit{edges of } G\} - \#\{\textit{vertices of } G\} + 1.$$

Suppose C is reduced with irreducible components $\Gamma_1, ..., \Gamma_n$, then

$$\beta(G) = g_a(C) - \sum_{i=1}^{n} g_a(\Gamma_i),$$

where g_a denote for the arithmetic genus.

Proof. The first part is obvious. Now since C is an effective Cartier divisor in X_s, thus C can be viewed as a curve over $k(s)$. Therefore we have the exact sequence

$$0 \to \mathcal{O}_{X_s} \,/\, \mathcal{O}_{X_s}(-C) \to \oplus_{i=1}^{n} \mathcal{O}_{X_s} \,/\, \mathcal{O}_{X_s}(-\Gamma_i).$$

Denote $\mathcal{F}$ to be the sheaf of cokernel of this sequence, then $\mathcal{F}_x$ is non-zero if and only if $x \in \Gamma_i \cap \Gamma_j$ for some distinct i, j. Then (by definition, cf. [Liu02, Exercise 2.2.9]) is a skyscraper sheaf with support in the intersection points of C. The above lemma then implies $\dim_{k(s)} \mathcal{F}(X_s) = \mathrm{length}_{k(s)} \mathcal{F}(X_s) = \sum_{i<j} \Gamma_i \cdot \Gamma_j$. Then we have the following relation of the Euler-Poicaré characteristics

$$\sum_{i=1}^{n} \chi_{k(s)}(\mathcal{O}_{\Gamma_i}) = \chi_{k(s)}(\mathcal{O}_C) + \chi_{k(s)}(\mathcal{F}) = \chi_{k(s)}(\mathcal{O}_C) + \sum_{i<j} \Gamma_i \cdot \Gamma_j.$$

The last equation follows from that $\mathcal{F}$ has finite support, so its i-th cohomology is zero for all $i \geq 1$..

On the other hand, by the definition of G, one has

$$\beta(G) = \sum_{i<j} \Gamma_i \cdot \Gamma_j - n + 1.$$

Since the arithmetic genus g_a is defined to be 1 minus the Euler-Poicaré characteristic, therefore

$$\sum_{i<j} \Gamma_i \cdot \Gamma_j - n + 1 = \sum_{i=1}^{n} \chi_{k(s)}(\mathcal{O}_{\Gamma_i}) - n + \chi_{k(s)}(\mathcal{O}_C) + 1 = g_a(C) - \sum_{i=1}^{n} g_a(\Gamma_i).$$

□

Proposition 3.5. *Let $C > 0$ be a reduced connected vertical divisor on X. Suppose the irreducible components Γ of C all verify $K_{X/S} \cdot \Gamma = 0$, and that C does not contain all of the irreducible components of X_s. Let G be the dual graph of C, then G is of one of the following forms (the indices denote the number of the vertices):*

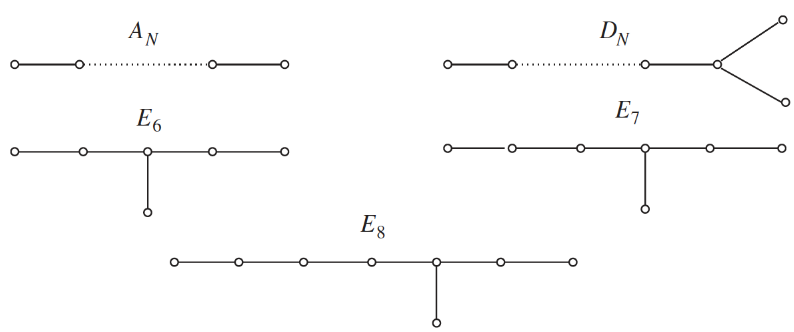

FIGURE 4. Classification of dual graphs

4. Weierstrass models of elliptic curves

Definition 4.1. *Let S be a scheme.*

1. *By a **curve** C over S, we mean $C \to S$ is a smooth morphism of relative dimension 1 (all the non-empty fibres are of dimension 1), which is separated and of finite presentation, i.e., locally of finite presentation, pullback of quasi-compact sets are quasi-compact, and the pull-back of quasi-compact sets in $C \times_S C$ under $C \to C \times_S C$ is quasi-compact.*
2. *By an **elliptic curve** E over S, we mean E is a proper smooth curve over S with geometrically connected fibres all of genus 1 and together with a fixed section 0.*

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ S \arrow[u, bend left, "0"] \end{tikzcd}$$

In order to be coherent to our notation above, we will now assume S to be an affine Dedekind scheme of dimension 1 and K be its field of fraction. Now for any elliptic curve E over K, one can apply the Riemann-Roch theorem to obtain a Weierstraß equation to E. We would like to generalise this notion to obtain a Weierstraß model for E over S.

Lemma 4.2. *Let $\pi : \mathcal{E} \to S$ be a fibred surface such that $\mathcal{E}_\eta = E$. Let o be the point given by $0 : \operatorname{Spec} K \to E$ and $O := \overline{\{o\}}$ be its closure in $\mathcal{E}$. Then*

1. *For any $n \geq 2$, $\mathcal{O}_{\mathcal{E}}(nO)$ is generated by its global sections.*
2. *The sheaf $\mathcal{L} := R^1\pi_* \mathcal{O}_{\mathcal{E}}$ is invertible on S. Suppose it is free, then for $n \geq 2$, there exists an exact sequence*
$$0 \to \pi_* \mathcal{O}_{\mathcal{E}}((n-1)O) \to \pi_* \mathcal{O}_{\mathcal{E}}(nO) \to \mathcal{L}^{\otimes n} \to 0,$$
$\pi_* \mathcal{O}_{\mathcal{E}}(nO)$ is free of rank n, and the canonical homomorphism
$$\oplus_{2a+3b\leq n}(\pi_* \mathcal{O}_{\mathcal{E}}(2O))^{\otimes a} \otimes (\pi_* \mathcal{O}_{\mathcal{E}}(3O)^{\otimes b}) \to \pi_* \mathcal{O}_{\mathcal{E}}(nO)$$
is surjective.

(cf. [Liu02, Lemma 9.4.29]*, see also* [Ka&Ma, Theorem 2.1.2]*)*

Proposition 4.3. *Assumption as above. Let $S = \operatorname{Spec} A$. Assume further that $\mathcal{E}_s$ is integral for all $s \in S$. Then*

1. *$\mathcal{E} \to S$ is local complete intersection and $\pi_*\omega_{\mathcal{E}/S}$ is an invertible sheaf.*
2. *Suppose $\pi_*\omega_{\mathcal{E}/S}$ is free over S. Then $\mathcal{E} \to S$ can be described by a integral Weierstraß equation. We then call $\mathcal{E}$ the Weierstraß model of E.*

(cf. [Liu02, Proposition 9.4.30]*, see also* [Ka&Ma, (2.2)]*)*

Idea of proof. We first assume the sheaf $\mathcal{L}$ in the previous Lemma is free on S and let $L(nO) = H^0(\mathcal{E}, \mathcal{O}_{\mathcal{E}}(nO))$. Then the Lemma implies there exists $x \in L(2O)$ and $y \in L(3O)$ such that $\{1, x\}$ is a basis of $L(2O)$ over A and $\{1, x, y\}$ is a basis of $L(3O)$ over A. Moreover, the images of x^3 and y^3 in $L(6O)/L(5O) \simeq H^1(\mathcal{E}, \mathcal{O}_{\mathcal{E}})^{\otimes 6}$ are both bases, therefore there exists $a_i \in A$ such that
$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$
This implies that the morphism $\varphi : \mathcal{E} \to \mathbf{P}^2_A$ associated to the basis $\{1, x, y\}$ of $L(3O)$ sends $\mathcal{E} - \{O\}$ into $\mathcal{E}'$ defined by the equation
$$Y^2Z + a_1XYZ + a_3Z^3Y = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$
Now it remains to verify that φ is an isomorphism.

For the general case $\mathcal{L}$ is locally free. The above shows that $\mathcal{E} \to S$ is a local complete intersection. Then by duality, one has $\pi_*\omega_{\mathcal{E}/S} \simeq (R^1\pi_* \mathcal{O}_{\mathcal{E}})^\vee = \mathcal{L}^\vee$, ans thus $\pi_*\omega_{\mathcal{E}/S}$ is invertible, and if it is free, $\mathcal{L}$ is also free. □

Proposition 4.4. *Let $\pi : \mathcal{E} \to S$ be the Weierstraß model as above. Then it is a normal fibred surface and*

1. *The morphism π is smooth at the points of O.*
2. *For any $s \in S$, $\mathcal{E}_s$ is geometrically integral.*

(3) *The morphism π is local complete intersection. Suppose the Weierstraß equation is given by*
$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$
Let
$$\omega := \frac{dx}{2y + a_1x + a_3} \in \Omega^1_{K(\mathcal{E})/K}.$$
Then $\omega_{\mathcal{E}/S} = \omega\,\mathcal{O}_{\mathcal{E}}$. In particular, $\pi_*\omega_{\mathcal{E}/S} = \omega\,\mathcal{O}_S$ is free on S.

(cf. [Liu02, Proposition 9.4.26]*, see also* [Ka&Ma, (2.2)]*)*

Remark. Now we have $\mathcal{E} \to S$ a fibred surface with the generic fibre isomorphic to E. Then there exists a relative minimal model of $\mathcal{E}$ by [Liu02, Proposition 9.3.19]. Since the genus of E is 1, thus [Liu02, Corollary 9.3.24] yields that such relative minimal model is minimal.

Theorem 4.5. *With the same notation as above. Let Δ denote the discriminant of the Weierstraß equation of $\mathcal{E}$. We suppose that $\mathcal{E}$ is the Weierstraß model amoung all such models of E such that the valuation of its discriminant at each point is the smallest among all Weierstraß models of E. Let $\rho : \mathcal{E}_{\min} \to S$ be the minimal regular model. Then*
$$\Xi := \{\textit{vertical prime divisors } \Gamma \textit{ of } \mathcal{E}_{\min} : \Gamma \cap O = \emptyset\}$$
is finite and there exists a contraction morphism $f : \mathcal{E}_{\min} \to \mathcal{E}$ of the divisors belong to Ξ. Moreover, $\mathcal{E}_{\min} \to S$ is locally complete intersection and one has $f_*\omega_{\mathcal{E}_{\min}/S} = \omega_{\mathcal{E}/S}$ and $\omega_{\mathcal{E}_{\min}/S} = f^*\omega_{\mathcal{E}/S}$. (cf. [Liu02, Theorem 9.4.35]*)*

Remark. Recall that the canonical model are defined for fibred surfaces which has generic fibres with genus ≥ 2. The above Theorem however tells us that the minimal Weierstraß model plays a role of canonical model in the case of genus-one curves, but instead of contracting -2-curves, we contract the prime divisors that do not intersect with O.

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