Homework Solutions.

1) $A = PDP^{-1}$, $A^4 = PD^4P^{-1}$. $D^4 = \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

$$A^4 = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$$

3) $$A^k = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} a^k & 0 \\ 2a^k & b^k \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^k & 0 \\ 2a^k - 2b^k & b^k \end{bmatrix}.$$

7) Char poly $= (\lambda+1)(\lambda-1)$, e-values $1, -1$, so is diagonalisable.

$\lambda = 1$: $$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix} \xrightarrow{\text{RRed}} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

$x_1 - \frac{x_2}{3} = 0$, $W_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$

$\lambda = -1$: $$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix} \xrightarrow{\text{RRed}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$W_2 = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}.$

$A = PDP^{-1}$, $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$

8) Char poly $= (\lambda-3)^2$, only e-value is 3.

e-vector: $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow[\text{R-red}]{} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$W_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, $\dim W_3 < 2$,

so no basis of e-vectors, so not diagonalisable.

25) No, A must be diagonalisable. The remaining e-value λ_3 must have $\dim W_{\lambda_3} \geq 1$,

so

$\sum_\lambda W_\lambda \geq 1+2+1 = 4$, so do have a basis of e-vectors.

13) $\text{dist}(\underline{x}, \underline{y}) = \text{length}(\underline{x}-\underline{y}) = \text{length}\left(\begin{bmatrix} 11 \\ 2 \end{bmatrix}\right)$

$= \sqrt{11^2 + 2^2} = \sqrt{121+4} = \sqrt{125} = 5\sqrt{5}.$

15) $a.b = \cancel{8} - 16 + 15 \neq 0$, not orthogonal.

16) $\underline{u}.\underline{v} = 24 - 9 - 15 = 0$, are orthogonal.

1) $u.u = 5$, $v.u = 8$, $\frac{v.u}{u.u} = \frac{8}{5}$

2) $w.w = 35$, $x.w = 5$, $\frac{x.w}{w.w} = \frac{5}{35} = \frac{1}{7}$

3) $w.w = 35$, $\frac{w}{w.w} = \begin{bmatrix} 3/35 \\ -1/35 \\ 5/35 \end{bmatrix}$

4) $\frac{u}{u.u} = \begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

5) $v.v = 50$ $\left(\frac{u.v}{v.v}\right)v = \frac{5}{50} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$

6) $x.x = 49$, $\left(\frac{x.w}{x.x}\right) \cdot x = \frac{5}{49}\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$

7) $\|w\| = \sqrt{w.w} = \sqrt{35}$

8) $\|8\| = \sqrt{x.x} = \sqrt{49} = 7.$

9) $\left\|\begin{bmatrix} -30 \\ 40 \end{bmatrix}\right\| = \sqrt{30^2 + 40^2} = 50$, so get $\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ for normalization.

10) $\left\|\begin{matrix} -6 \\ 4 \\ -3 \end{matrix}\right\| = \sqrt{61}$, so normalisation in $\begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}.$