Eigenvalues and Eigenvectors
So far:

- Definitions:

  \[ M \text{ an } n \times n \text{ matrix over field } k = \mathbb{R} \]

  \[ \text{Eigenvector: nonzero } v \in k^n \text{ such that there exists } \lambda \in k \text{ with } Mv = \lambda v \]

  \[ \text{Eigenvalue: } \lambda \in k \text{ such that there exists nonzero } v \in k^n \text{ with } Mv = \lambda v \]

  \[ \text{Eigenspace: } W_\lambda \subseteq k^n, \text{ the set of eigenvectors with a given eigenvalue } \lambda \text{ (linear subspace of } k^n) \]

- Computation:

  Eigenvalues are roots in \( k \) of characteristic polynomial:
  \[ \det(M - \lambda I_n) \]

  To find eigenvectors for eigenvalue \( \lambda \), solve the matrix equation \( Mv = \lambda v \) (Gaussian elimination/row reduction)
Why? Fish in a pond

- In year $n$ we have $a$ adult and $b$ baby fish

- In year $n + 1$ we have
  
  $0.9a + 0.2b$ adults
  $0.5a + 0.3b$ babies

- Let $M = \begin{bmatrix} 0.9 & 0.2 \\ 0.5 & 0.3 \end{bmatrix}$

- Population modelled by linear function
  
  $T: \mathbb{R}^2 \to \mathbb{R}^2$
  
  $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto M \begin{bmatrix} a \\ b \end{bmatrix}$

- Eigenvalues and vectors of $M$ (approximate):
  
  $\lambda_1 = 1.04$, $v_1 = \begin{bmatrix} 83 \\ 56 \end{bmatrix}$
  
  $\lambda_2 = 0.16$, $v_2 = \begin{bmatrix} -26 \\ 96 \end{bmatrix}$
What happens in long term?

• Suppose the population is
  \[ a = 83 \]
  \[ b = 56 \]

• The population vector is (roughly) an eigenvector with eigenvalue
  \[ \lambda_1 = 1.04 \]

• So after \( r \) years, the population will be
  \[ M^r \begin{bmatrix} 83 \\ 56 \end{bmatrix} = M^{r-1} \lambda_1 \begin{bmatrix} 83 \\ 56 \end{bmatrix} = \cdots = \lambda_1^r \begin{bmatrix} 83 \\ 56 \end{bmatrix} = 1.04^r \begin{bmatrix} 83 \\ 56 \end{bmatrix} \]

• Grows slowly over time.
Graphical representation:

Stable population, slowly growing
Graphical representation:

\[ \lambda = 0.16 \]

\[ \lambda = 1.04 \]

Converges to ratio \( a/b = 83/56 \)
Effect of pollution

• Suppose some pollution is introduced to our pond, making fewer babies grow to adulthood

• New matrix

\[
M' = \begin{bmatrix}
0.9 & 0.1 \\
0.5 & 0.3
\end{bmatrix}
\]

• Eigenvectors *roughly* the same

• \(\lambda_2 \approx 0.16\), *roughly* unchanged

• \(\lambda_1 = 0.97\), slightly smaller (was 1.04)

\[
(M')^r \begin{bmatrix} 83 \\ 56 \end{bmatrix} = \cdots = 0.97^r \begin{bmatrix} 83 \\ 56 \end{bmatrix}, \text{ slowly goes to zero}
\]
Graphical representation (polluted): 

Population always collapses to zero!
Graphical representation (polluted):

\[ \lambda = 0.16 \]

\[ \lambda = 0.97 \]

Population always collapses to zero!
Conclusion

Take-away:

• small changes in parameters can have a big effect on long-term behaviour of linear models

• eigenvalues and vectors are a powerful tool to detect this

• especially important if we have more variables (male and female fish?)

Next:

• diagonalisation using eigenvalues and vectors

• using diagonalisation to compute large powers of a matrix

• computing the population of our pond in 1000 years, fast