1. Consider the matrix \( A \) given by
\[
A = \begin{bmatrix}
3 & 0 & -3 & 0 \\
4 & -1 & -1 & 2 \\
-1 & 1 & -2 & -2
\end{bmatrix}
\]
a) Put the matrix \( A \) in reduced echelon form. 
\text{Hint: your answer should be one of the following two options:}
\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -3 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -3 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
b) Write the solution set of the homogeneous linear system \( Ax = 0 \) as a span of vectors.
c) What is the dimension of the solution set?

2. We define several vectors in \( \mathbb{R}^3 \):
\[
a = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad c = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.
\]
a) Is the set \( \{ a, b \} \) orthogonal?
b) Is the set \( \{ a, c \} \) orthogonal?
c) Is the set \( \{ a, b, c \} \) orthogonal?
d) Compute the projection of the vector \( d \) onto the span of \( \{ a, b \} \).

3. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.
a) For every pair \( A, B \) of \( n \times n \) matrices, we have that \((AB)^T = A^T B^T\) (where \( M^T \) denotes the transpose of \( M \)).
b) Let \( A \) be an \( n \times n \) matrix, and let \( B \) be the matrix obtained from \( A \) by multiplying every entry of \( A \) by 2. Then \( \det B = 2^n \det A \).
c) If \( U \) and \( V \) are linear subspaces of \( \mathbb{R}^n \), then the intersection \( U \cap V \) is a linear subspace of \( \mathbb{R}^n \).
d) If \( \lambda \) is an eigenvalue of an \( n \times n \) matrix \( A \), then the set of eigenvectors with eigenvalue \( \lambda \) is equal to the set of non-zero solutions of the matrix equation \((A - \lambda I_n)x = 0\).
e) If \( a, b \) and \( c \) are non-zero vectors in \( \mathbb{R}^2 \) with \( a \) orthogonal to \( b \) and \( b \) orthogonal to \( c \), then it must hold that \( a \) is a scalar multiple of \( c \).
4. We define matrices $A$ and $B$ by

\[ A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

In the remainder of this question you may use without proof that $A$ and $B$ are row equivalent.

a) Give a basis for the null space $\text{Null}(A)$.

b) Give a basis for the column space $\text{Col}(A)$.

Now let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the linear transformation given by $T(x) = Ax$.

c) State the definition of the image of a linear transformation.

d) Write down a basis of the image $\text{im}(T)$.

5. We define a matrix $A$ by

\[ A = \begin{bmatrix} 12 & 5 \\ -30 & -13 \end{bmatrix}. \]

a) Show that 2 and $-3$ are eigenvalues of $A$.

b) Find a basis for the eigenspace for 2.

c) Find a basis for the eigenspace for $-3$.

d) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $PDP^{-1} = A$.

e) Show that $A^{2019} = \begin{bmatrix} 6(2^{2018} + 3^{2018}) & 2^{2019} + 3^{2019} \\ -6(2^{2019} + 3^{2019}) & -(2^{2020} + 3^{2020}) \end{bmatrix}$.

6. We define a matrix $A$ and a vector $b$ by

\[ A = \begin{bmatrix} -2 & 1 \\ -2 & 1 \\ 3 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}. \]

(a) Compute the product $A^T A$.

(b) Compute the product $A^T b$.

(c) Compute the least squares solution to the equation $Ax = b$. Hint: the entries of your answer should be integers.

An experiment gave the datapoints: $(1, 1), (1, 0), (4, 1), (4, 4)$. We want to find the best-fitting curve of the form

\[ y = v_0 + v_1 \sqrt{x} \]

(here we only allow $x$ to take positive values, and $\sqrt{x}$ means the positive square root).

(d) Write a matrix equation whose solution is the vector $\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}$ giving the curve which best fits the data.