Fibrations

David Holmes

Abstract

Note: this is not a masters dissertation; due to the nature of the British university system, I have not undertaken a masters. The nearest equivalent to a masters dissertation I have produced is this essay, which I submitted as part of my Certificate of Advanced Studies in Mathematics in Cambridge.

The starting point for this essay was the papers [3] of Campana, [10] of Kebekus and Kovacs, [12] of Viehweg and [7] of Lu. Its main aims are:

1. To clarify and expand some of the introductory material from [3] to make it more accessible to beginning algebraic geometers,

and

2. To apply some of the techniques of these papers to the case of fibrations from a surface to $\mathbb{P}^1$, with particular interest in the cases when

- the Kodaira dimension of the surface is non-negative
- the genus of the fibres is greater than 1.

The main results of this are:

• Expanded proofs of several of Campana's foundational results.

• A simple proof that fibrations over smooth curves are always admissible.

• A (to the best of my knowledge new) proof (Proposition 5) that the Kodaira dimension of a fibration from a blowup of $\mathbb{P}^2$ to $\mathbb{P}^1$ as defined in section 2 is negative.

•A family of such fibrations with $\Delta(f)$ arbitrarily close to its maximum possible value.

• A table (in Section 3.4) of upper bounds for the Kodaira dimension of fibrations to $\mathbb{P}^1$ from surfaces embedded in $\mathbb{P}^1 \times \mathbb{P}^2$, and a family of examples (in Section 3.7) showing these can be obtained for surfaces of high enough degree.

• An investigation into obstructions to extending one of Campana's examples to the case of a prepared divisor (section 5).

Thanks to Caucher Birkar for supervising this essay, and to Stephen Coughlan, Sarah Davis, Alvaro Nolla and Miles Reid for introducing me to the wonderful world of algebraic geometry.

Contents

1 Definitions and basic theorems for working with fibrations

We work always over the field of complex numbers ($\mathbb{C}$) unless otherwise stated.

A scheme is a quasiprojective scheme.

A variety is a reduced and irreducible scheme.

For a projective scheme X, K_X denotes the canonical divisor class of X.

When we blow up a surface, we always blow up at a point, not along a curve.

1.1 Motivating examples

1.1.1 A multiple fibre

Let $X : (x^2 = y) \subset \mathbb{A}^2_{(x:y:z)}$, let Y be the line $x = 0$, and let $f : X \to Y$ be the projection. Then we note that the fibre X_y consists of a pair of distinct points (each with multiplicity 1) if $y \neq 0$, but that the fibre over 0 is a double point, and as such is not smooth as a scheme (though both X and Y are).

1.1.2 A pair of elliptic fibrations

This example is taken from [3].

Let $X_1 := E \times \mathbb{P}^1$, where E is an elliptic curve. Then X_1 has a natural fibration to $\mathbb{P}^1$, with all fibres isomorphic to E, so smooth as schemes.

Let H be a hyperelliptic curve, so H has a degree 2 morphism φ to $\mathbb{P}^1$. Let h be the hyperelliptic involution on H (so $\frac{H}{<h>} \cong \mathbb{P}^1$), and let t be a translation of E of order 2.

Let $X := (E \times H)/\langle t \times h\rangle$. This has a natural fibration to $\mathbb{P}^1$ via φ.

Then it is easy to see that as reduced schemes, all the fibres of this map are isomorphic to E, and so are smooth. However, if we let $p \in \mathbb{P}^1$ be a point such that $\varphi^{-1}(p)$ is not smooth as a scheme (it is a double point), we find that the corresponding fibre X_p also has multiplicity 2, and so is not smooth as a scheme (the corresponding reduced scheme is isomorphic to E.).

As we progress, we will define various invariants of fibrations, which we will apply to these examples to reveal how different they are, even though they are both fibrations to $\mathbb{P}^1$ with their (reduced) fibres smooth and isomorphic to E.

1.2 What is a fibration?

(These definitions are based on [3].)

Above we refered to some of our maps as *fibrations*, so we should define this.

Definition 1 (Fibration) *Let X, Y be two normal (eg. smooth in codimension 1) complex connected projective varieties (reduced and irreducible). A* fibration *is a dominant meromorphic map*

$$f : X \to Y \tag{1}$$

with connected fibres.

Such a fibration is call holomorphic *if the map f is.*

We call X the top *space, and Y the* base *space.*

We say the fibration is smooth *if both X and Y are smooth (NOT if f is smooth in the sense that all fibres are smooth - such fibrations are not particularly interesting for us, as their multiplicity divisor will always be 0 (see below)).*

Thus we can see that the two examples in Section 1.1.2 are fibrations, wheras the map in 1.1.1 is not, both because X and Y are not projective, and because the fibres are not connected.

We say two fibrations $f : X \to Y$ and $f' : X' \to Y'$ are *equivalent* if there exist bimeromorphic maps $u : X \leftrightarrow X'$ and $v : Y \leftrightarrow Y'$ such that

$$\begin{array}{ccccc} & X & \overset{u}{\longleftrightarrow} & X' & \\ f & \downarrow & \Box & \downarrow & f' \\ & Y & \overset{v}{\longleftrightarrow} & Y' & \end{array} \tag{2}$$

commutes.

This is clearly an equivalence relation.

We can also define another relation on fibrations, given by *dominance*:

Definition 2 (Dominance) *Given two fibrations $f : X \to Y$ and $f' : X' \to Y'$, we say f* dominates *f' if there exist* holomorphic *maps $u : X \to X'$ and $v : Y \to Y'$ with meromorphic inverses such that*

$$\begin{array}{ccccc} & X & \overset{u}{\to} & X' & \\ f & \downarrow & & \downarrow & f' \\ & Y & \overset{v}{\to} & Y' & \end{array} \tag{3}$$

commutes.

It is easy to see that this is reflexive and transitive, and it is antisymmetric on (biholomorphic) isomorphism classes[1] of spaces, so it is a partial ordering on isomorphism classes.

1.3 Orbifolds

(This definition is based on [3]).

We saw in Section 1.1.2 that it is possible to have two fibrations with the same reduced fibres and the same base, but with very different structures. Part of the way we shall describe this difference is by viewing the base as an *orbifold* rather than simply as a variety.

[1] ie classes of fibrations which are equivalent as defined above, but where the maps u and v are biholomorphic instead of bimeromorphic

Definition 3 (Orbifold) *An orbifold is a pair (X/Δ) where X is a variety and Δ is a Weil $\mathbb{Q}$-divisor on X.*

[3] gives an example to show the connection between this definition and the classical view of an orbifold as being locally a quotient of a manifold. Orbifolds are in some way a natural generalisation of varieties - many concepts defined on varieties extend to orbifolds, such as Riemann-Roch (see [2]), however for the purposes of this essay, it is adequet just to think of them as a pair of a variety and a divisor as above.

1.4 Multiplicity of fibres

(These definitions are based on [3]).

Recall from the examples above that we are interested in the **multiplicity** of the fibres of a fibration. We can describe this well by putting an orbifold structure on the base Y as follows:

Let $S \subset Y$ be the set of points $s \in Y$ such that the scheme-theoretic fibre over s is not smooth (clearly if the fibre over s is multiple, this will be the case). Then for any codimension 1 subvariety $V$of Y such that $V \subset S$, write

$$f^*(V) = \sum_D m_D.D + R \tag{4}$$

where the sum runs over all irreducible components D of the fibre over V which map *surjectively* onto V, recording their multiplicity by m_D (and so R is the largest divisor in $f^*(V)$ the support of which does not surject onto V).

Then define m_V the **multiplicity of the fibre over** V to be

$$m_V := \inf_D m_D \tag{5}$$

where again the infinitum runs over all irreducible components D of the fibre over V which map surjectively onto V.

We then define m_V to be 1 for all codimension 1 subvarieties $V \subset Y$ which are not contained in S.

Then we can define the **multiplicity divisor of the fibration** to be:

$$\Delta(f) := \sum_V (1 - 1/m_V).V \tag{6}$$

where the sum runs over all codimension 1 subvarieties of Y.

Then we say the **orbifold base of the fibration** f is the pair $(X/\Delta(f))$.

For example, we can then see that in Section 1.1.2, the orbifold base of the first fibration is $(\mathbb{P}^1/0)$ wheras for the second it depends on the genus of the hyperelliptic curve C; the orbifold base is $(\mathbb{P}^1/g(C).(1-\frac{1}{2}).[point])$ (by considering the canonical map to $\mathbb{P}^1$).

1.5 Kodaira dimension

We recall the definition of the *Kodiara dimension* of a pair (X/Δ) where Δ is a $\mathbb{Q}$ divisor on X (based on [1]):

Definition 4 **($\kappa(X/\Delta)$)** *Fix a pair (X/Δ) where Δ is a $\mathbb{Q}$ divisor on X. Let $n \in \mathbb{Z}_{>0}$ such that $n.(K_X + \Delta)$ is an integral $\mathbb{Z}$-divisor on X. If for every $r > 0$ we get that $h^0(X, \mathcal{O}(r.n.(K_X + \Delta)) = 0$, then we set the Kodiara dimension $\kappa(X/\Delta)$ to be $-\infty$.*

Otherwise, Given $r \in \mathbb{Z}_{>0}$, we get a map

$$\begin{array}{rccl} \varphi_r : & X & \to & \mathbb{P}\left(H^0(X, \mathcal{O}(r.n.(K_X + \Delta))^*\right) \\ & p & \mapsto & (evaluation\ at\ p). \end{array} \tag{7}$$

Then we let

$$\kappa(X/\Delta) := \max_{r \in \mathbb{N}} \left(\dim_{\mathbb{C}} \left(\varphi_r(X)\right)\right) \tag{8}$$

be the Kodaira dimension of (X/Δ)

Then we set the Kodaira dimension of a variety X to be

$$\kappa(X) := \kappa(X/0) \tag{9}$$

Looking at the examples in Section 1.1.2, the orbifold base of the first fibration is $(\mathbb{P}^1/0)$ and $\kappa(\mathbb{P}^1/0) = -\infty$. In contrast, for the second example the base is $(\mathbb{P}^1/g.(1 - \frac{1}{2}).[point])$ and

$$\kappa\left(\mathbb{P}^1, K_{\mathbb{P}^1} + g.(1 - \frac{1}{2}).[point]\right) = \begin{cases} -\infty & \text{if } g \leq 3 \\ 0 & \text{if } g = 4 \\ 1 & \text{if } g \geq 5 \end{cases}$$

We can also calculate the Kodaira dimensions of the top spaces of these fibrations; $K_{\mathbb{P}^1}$ has no holomorphic global sections, so the Kodaira dimension of the top space in the first example is $-\infty$. In contrast, K_E always has constant global sections, and C has a $g(C)$ dimensional space of holomorphic global sections, so the Kodaira dimension of the top space in the second example is 1.

1.6 Kodaira dimension of a fibration

We could now simply define the Kodiara dimension of a fibration to be the Kodiara dimension of its orbifold base. However, as we will define in Section 1.7, and see throughout the calculations which form the bulk of this essay, fibrations are often better thought of as "embeddings" of Y in the *Chow scheme* of X, which depend only on the equivalence class of the fibration. If we are really thinking of equivalence classes of fibrations, it makes sense to give a definition which works on equivalence classes. Thus we define the Kodiara dimension of a fibration to be

$$\kappa(f) := \inf \kappa(Y'/\Delta(f')) \tag{10}$$

as $f' : X' \to Y'$ runs over all *holomorphic* fibrations between *smooth* X', Y' equivalent to f.

We say that a fibration $f : X \to Y$ is *admissible* if $\kappa(f) = \kappa(Y/\Delta(f))$. Determining whether a given fibration is admissible is an interesting problem; we will show quite easily in Section 1.8 that any fibration to a curve is admissible, as well as a few other special cases. The main result on this is that *neat* fibrations are admissible (see later for a definition), which implies that all fibrations to curves are admissible. However, the proof of this is quite technical, and admissibility over curves is all we will need for our calculations, so we leave it to Section 6.

To show this definition of Kodaira dimension makes sense, we need to show that any (even meromorphic) fibration has a model with X and Y smooth, and f holomorphic. In fact we show the stronger result that we can always find such a model *dominating* f.

Proposition 1 *Let $f : X \dashrightarrow Y$ be a meromorphic fibration. Then there exists a holomorphic fibration $f' : X' \to Y'$ dominating f between smooth complex spaces X' and Y'.*

Proof:

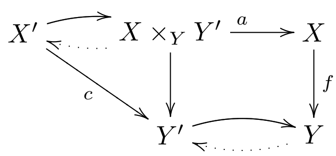
We will use Hironaka's resolution of singularities, a readable account of which can be found in [6]. The weakest version of his theorem (which is sufficient for our purposes here) is:

Proposition 2 (Weakest resolution in characteristic zero) *Given a projective variety X over a field of characteristic zero, there exists a smooth projective variety X' and a proper morphism $g : X' \to X$ such that g is bimeromorphic.*

We omitt the proof of this result because it is hard! (The result can be found on page 119 of [6], note that in [6] varieties are always quasiprojective.)

We know (see [4]) that we can take a blowup $b : \overline{X} \to X$ such that $b \circ f$ is holomorphic, so we may assume f is holomorphic.

Then by Proposition 2 we can take Y' smooth with a proper birational holomorphic map to Y. We can then take the fiber product $X \times_Y Y'$ and take X' smooth with a proper bimeromorphic holomorphic map to $X \times_Y Y'$ as in the diagram below:



where the map c is obtained by composition. Thus it remains only to show that the map $a : X \times_Y Y' \to X$ has a meromorphic inverse. For this we look at the function fields of the varieties involved in the fibre product, and use its universal property. We work on an affine piece, and so we can write :

$X = Spec(A)$

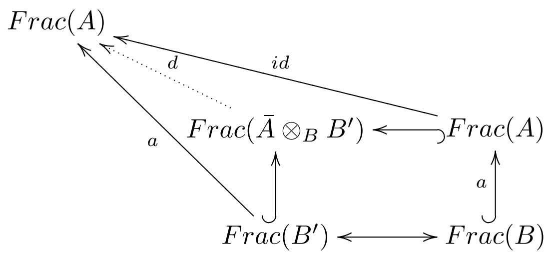
$Y = Spec(B)$

and denote the corresponding rings for: X' as A'

and

Y' as B'.

Then the relevant diagram for the universal property with respect to $Frac(\overline{A})$ of the fibre product is below:



recalling that a non-zero map between fields is an injection, and it is easy to see the maps are non zero. The unversal property gives us the existance of the map d, and so we are done. □

It is interesting to note that the proof would not simplify significantly if we restrict to the case where Y is a curve; we would have Y' isomorphic to Y, but the meat of the proof was about constructing the correct top space X'.

We will see later that any fibration to a curve is admissible. Hence we have that for the examples in Section 1.1.2, the Kodiara dimension of the first fibration $\kappa(f) = \kappa(\mathbb{P}^1/0) = -\infty$, wheras for the second we have

$$\kappa(f) = \begin{cases} -\infty & \text{if } g(C) \leq 3 \\ 0 & \text{if } g(C) = 4 \\ 1 & \text{if } g(C) \geq 5 \end{cases}$$

where C is the hyperelliptic curve.

Not all holomorphic fibrations between smooth spaces are admissible; for an example of this, see Section 5.

1.7 Chow schemes and Fibrations

We can view a holomorphic fibration $f : X \to Y$ as an embedding $Y \hookrightarrow \mathcal{C}(X)$ (or correspondingly a meromorphic fibration as a subset $Y' \subset \mathcal{C}(X)$ bimeromorphic to Y), where $\mathcal{C}(X)$ is the *Chow scheme* of X. As we will see later, this is a very useful technique for computation, as well as for theoretical purposes as seen in [3].

The full definition of the Chow scheme of a general scheme is quite complicated (see [5]) and is not needed for our purposes, so we will not give details. All the schemes we will use in our calculations will be projective varieties over $\mathbb{C}$, so there is a much simpler embedded construction we will use.

1.7.1 What is a Chow scheme?

Proofs and basic definitions in this section are omitted for reasons of time and relevance, see [5].

The Chow scheme is a scheme which parametrises algebraic cycles in a given variety. Much as the Hilbert scheme parametrises subschemes of given Hilbert polynomial, we set $\mathcal{C}(X)^n_d$ to be the scheme parametrising algebraic cycles of degree d and pure dimension n [2], then the Chow scheme $\mathcal{C}(X)$ of X is

$$\mathcal{C}(X) := \coprod_{n,d\in\mathbb{Z}} \mathcal{C}(X)^n_d. \tag{11}$$

There are two main ways to construct the Chow scheme for a given projective variety; one is at the scheme representing a given functor (as for Hilbert schemes), the other is a more concrete construction as a subscheme of a larger projective space. We will use both, but will not prove that they are equivalent. We will use the functorial approach for proving things (it can be defined for any scheme, though we will not do this), but the projective approach for our calculations later because it is much easier to work with.

1.7.2 The Chow scheme as the unique scheme representing a functor

Fix a projective variety X over $\mathbb{C}$, fix $n \geq 0$, d any integer. It is easy to give a *set* parametrising alg cycles of dimension n and degree d in X. The challenge is to put a geometric structure on it, i.e. to make it a scheme which 'smoothly' parametrises cycles. The notion of 'smoothly parametrising' that we want is essentially given by properness, though several other conditions are needed.

First we construct the relevant functor F from the category of normal k-schemes to the category of sets sending a normal k-scheme S to the set of closed subschemes $\Sigma \hookrightarrow X \times S$ such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma & \hookrightarrow & X \times S \\ & {\scriptstyle\pi}\searrow & \downarrow {\scriptstyle pr_2} \\ & & S \end{array}$$

where pr_2 is projection to the second factor, π is 'nice' and for every s in S, the fibre $\pi^{-1}(s) \hookrightarrow X$ is a closed subscheme of dimension n and degree d. Here 'niceness' is quite technical to define, and we refer that reader to [5] for details. (F of a morphism is easy to see). We omitt the proof that F is a contravariant functor.

For F to be representable means that there is a unique k-scheme $\mathcal{C}$ such that for any k-scheme S,

$$F(S) = Mor(S, \mathcal{C}) \tag{12}$$

We omitt the proof that F is representable. Assuming then that F is representable, I claim (again without proof) that $\mathcal{C}$ is the scheme we want parametrising cycles in X of dimension n and degree d. To see why this is reasonable, consider some simple cases:

1. S is a point. Then $Mor(S, \mathcal{C})$ is just picking out single cycles of dimension n and degree d.

2. S is a pair of points. Then $Mor(S, \mathcal{C})$ is just picking out pairs of cycles of dimension n and degree d.

3. S is $\mathbb{P}^1_k$. Then $Mor(S, \mathcal{C})$ is just picking out nice families of cycles of dimension n and degree d parametrised by $\mathbb{P}^1$.

4. S is any k-scheme. Then $Mor(S, \mathcal{C})$ is picking out nice families of cycles of dimension n and degree d parametrised by S.

[2] ie cycles $C = \Sigma_i r_i Z_i$ with r_i integers, Z_i closed subvarieties all of dimension n, and $\Sigma_i r_i . \deg(Z_i) = d$.

1.7.3 The Chow scheme inside projective space

Let $X \hookrightarrow \mathbb{P}^N$ be a projective variety, fix $n \geq 0, d \in \mathbb{Z}$. Clearly cycles of X of dimension n and degree d have the same dimension and degree in $\mathbb{P}^N$, so we can start by parametrising cycles of $\mathbb{P}^N$ of dimension n and degree d, and then view $\mathcal{C}(X)^n_d$ inside this scheme.

Cycles are sums of subvarieties, so we start by defining a space parametrising subvarieties; let

$$\mathcal{C}h_0 : (\text{subvarieties of dimension } n\ ) \to Div(((\mathbb{P}^N)^*)^{N-n})$$

sending a subvariety $Z \subset \mathbb{P}^N$ to the set of all $N-n$-tuples of hyperplanes $H_1, \dots, H_{N-n}$ in $\mathbb{P}^N$ such that $Z \cap H_1 \cap \ldots \cap H_{N-n} \neq \emptyset$. To see this defines a divisor, view it as the (proper) projection to $((\mathbb{P}^N)^*)^{N-n}$ of the incidence graph

$$Z \hookrightarrow \{(p, H) : p \in H\} \subset Div(\mathbb{P}^N \times ((\mathbb{P}^N)^*)^{N-n})$$

This does not seem to have achieved much - we have simply sent a codimension n subvariety of $\mathbb{P}^N$ to a codimension 1 subvariety of a product of projective spaces. However, the key point is that a divisor (on a smooth space, eg projective space) is locally principle, and hence there is a bijection

$$\mathcal{C}h : \left(\text{cycles on } \mathbb{P}^N\right) \leftrightarrow ((\mathbb{P}^N)^*)^{N-n}$$

sending $d.Z$ to a divisor defined by the d-th power of the defining equation of $\mathcal{C}h(Z)$, and with the image of a sum defined in a similar way.

It is then easy to understand $Div(((\mathbb{P}^N)^*)^{N-n})$ as a product $N-n$ copies of projective spaces of dimension $\binom{N+d}{d} - 1$ (this can be seen by considering the dimensions of the spaces of forms of degree d in $N+1$ variables).

It is easy to see $\mathcal{C}(X)^n_d$ inside this scheme.

1.7.4 Viewing fibrations inside Chow schemes

Given a holomorphic fibration $f : X \to Y$, we define a map $Y \to \mathcal{C}h(X)$ sending $y \in Y$ to the cycle corresponding to the reduced fibre over y. Clearly this map is an injection, and we get an inverse map back to Y just given by f - we can view this map as the projection back to Y of the incidence graph as above. Changing to an equivalent fibration $f' : X' \to Y'$ gives $\mathcal{C}h(X') \to \mathcal{C}h(X)$ and the corresponding subset $Y' \hookrightarrow \mathcal{C}h(X')$ is bimeromorphic to $Y \hookrightarrow \mathcal{C}h(X)$. Hence we get a bijection between equivalence classs of holomorphic fibrations and compact irreducible analytic subsets of $\mathcal{C}h(X)$ with incidence graph bimeromorphic to Y. (This can easily be extended to meromorphic fibrations, but we will not need this). To clarify, we give a simple example:

$$X = \mathbb{P}^1 \times \mathbb{P}^1, Y = \mathbb{P}^1$$

with f the projection to the first factor. Then $\mathcal{C}h^1_1(X) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the map $Y \hookrightarrow \mathcal{C}h^1_1(X)$ is just embedding as a line, clearly biholomorphic to Y. Supposing we blow up X at a point $(0:1)(0:1)$ to give X', and take the corresponding fibration $f : X' \to \mathbb{P}^1 = Y'$. Then $\mathcal{C}h^1_1(X')$ has more than one irreducible component; there is the component corresponding to the principal divisors, and then the component for the exceptional divisor. The part of $\mathcal{C}h_1^1(X')$ which contains the fibres of f' is that of principal divisors; the fibres of f' are the proper transforms under the blowup of the fibres of f. That component of $\mathcal{C}h_1^1(X')$ is clearly the same as $\mathcal{C}h_1^1(X) \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $Y \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ as a line as before.

1.8 Admissibility over curves

The main aim of this section is to prove the following result which we will need for our calculations in later sections:

Let $f : X \to C$ be a holomorphic fibration from a smooth projective variety to a smooth curve.
Then $\kappa(f) = \kappa(C/\Delta(f))$, ie f is admissible.

Along the way we will prove several other results which do not have immediate applications to our calculations, but are of general interest. As such, some of the proofs could be somewhat shortened by restricting to the case of fibrations over a curve, but we will not do this in general.

1.8.1 Invariance of $\kappa(f)$ under birational changes of the top space over a curve

Next we introduce a stronger result for fibrations of smooth projective varieties over smooth curves, the proof of which is origional to the best of my knowledge.

Proposition 3 *Let X, X' be smooth projective varieties and C a smooth curve.*
Let $f : X \to C$ and $f' : X' \to C$ be two equivalent holomorphic fibrations, so we have a bimeromorphic map $u : X \dashrightarrow X'$ such that

$$\begin{array}{ccccc} & X & \overset{u}{\dashrightarrow} & X' & \\ f & \downarrow & & \downarrow & f' \\ & C & \overset{identity}{\leftrightarrow} & C & \end{array} \tag{13}$$

commutes. Then we have:

$$\Delta(f \circ u) = \Delta(f) \tag{14}$$

and hence

$$\kappa(Y/\Delta(f \circ u)) = \kappa(Y/\Delta(f)). \tag{15}$$

Note that this differs from Proposition 7 in that we do not require that u is holomorphic.

Note that in working over curves we have the advantages that:
1.Any component that is mappped to a point surjects onto it.
2. Each irreducible component of the fibre has dimension exactly $\dim X - 1$, which is particularly useful since any birational map of smooth projective varieties is defined outside a subset of codimension 2.

Proof:

Fix V in Y a codimension 1 subvariety (ie. a single point with multiplcity 1).

Write

$$f^*(V) = \sum_{D \subset X} m_D.D$$

and

$$f'^*(V) = \sum_{E \subset X'} m_E.E$$

where the sums run over all codimension 1 subvarieties of X (or X') mapped to V.

Then given a codimension 1 subvariety E of X', we define $u^*(E)$ to be the closure of the pullback of E intersect the domain of definition of u^{-1}. Similarly for u^{-1}. These are well defined since u is defined outside a codimension 2 subvariety.

Then we see that

$$\sum_E m_E.u^*(E) \leq \sum_D m_D.D$$

and similarly for $(u^{-1})^*$. Then by symmetry and the fact that u^* and $(u^{-1})^*$ preserve multiplicity on smooth varieties, we are done.

□

1.8.2 Admissibility of of smooth holomorphic fibrations over curves

The results above combine to give a corollary which will be very useful in later sections, and is the main useful result of this section for us:

Proposition 4 *Let $f : X \to C$ be a holomorphic fibration from a smooth projective variety over $\mathbb{C}$ to a smooth curve.*

Then $\kappa(f) = \kappa(C/\Delta(f))$, ie f is admissible.

Proof:

We start off by taking a smooth holomorphic admissible model $f' : X' \to C'$ of $f : X \to C$, possible by the definition on the Kodaira dimension.

We note as C is a smooth curve, any birational map from C to another smooth curve is an isomorphism, and thus $C \cong C'$.

Then by Proposition 3, we are done. □

2 Fibrations from blowups of $\mathbb{P}^2$ to $\mathbb{P}^1$

This is the first class of fibrations to $\mathbb{P}^1$ we will consider. They are simple in that the Kodaira dimension $\kappa(X)$ is always negative. However, they provide a good starting point before looking at more interesting fibrations.

2.1 The space of degree d forms on $\mathbb{P}^2$ vanishing at n points

We refer to [8] for basic results about such spaces.

Let $S_d^{\Phi_n}$ denote the vector space of forms of degree d vanishing on some set Φ_n of points of $\mathbb{P}^2$, where $\#\Phi_n = n$. By abuse of notation we will often write S_d^n, where we are assuming that the elements of Φ_n are in general position [3]. We may write S_d for $S_d^{\emptyset}$.

It is easy to see that

$$\dim_{\mathbb{C}}(S_d^{\emptyset}) = \binom{d+2}{2}. \tag{16}$$

Further, each element of Φ_n introduces one linear condition on $S_d^{\emptyset}$, so we have:

$$\dim_{\mathbb{C}}(S_d^{\Phi_n}) \geq \binom{d+2}{2} - n \tag{17}$$

where equality holds if Φ_n is a single point, or a pair of distinct points.

2.2 The map to $\mathbb{P}^1$

If $\dim_{\mathbb{C}}(S_d^{\Phi_n}) = 2$, there is an obvious way to define a rational map from $\mathbb{P}^2$ to $\mathbb{P}^1 \cong \mathbb{P}(S_d^{\Phi_n})$, by choosing a basis of $S_d^{\Phi_n}$.

In a coordinate free formulation, we can describe this map as follows:

Start with

$$S_d^{\Phi_n} \hookrightarrow S_d \tag{18}$$

as a linear subspace. Then we can define the corresponding map on the duals

$$(S_d)^* \to (S_d^{\Phi_n})^* \tag{19}$$

and hence on their projectivisations (recalling $\mathbb{P}(V^*) = \mathbb{P}(V)^*$)

$$g : \mathbb{P}(S_d)^* \dashrightarrow \mathbb{P}(S_d^{\Phi_n})^*. \tag{20}$$

which is well defined outside the projectivisation of the codimension 2 linear subspace of $(S_d)^*$ sitting over 0 in $(S_d^{\Phi_n})^*$. We also have a morphism

$$\begin{aligned} h : \mathbb{P}^2 &\to \mathbb{P}(S_d)^* \\ p &\mapsto (\textit{evaluation at } p). \end{aligned} \tag{21}$$

We wish then to define a map $g \circ h : \mathbb{P}^2 \dashrightarrow \mathbb{P}(S_d^{\Phi_n})^*$ by composition, but for this we need that the intersection of the domain of definition of g with the image of h is dense open in the image of h. However, this is clear since the domain of definition of g is dense open in $\mathbb{P}(S_d)^*$, and its intersection with the image of h is nonempty (otherwise all elements of $S_d^{\Phi_n}$ vanish on the whole of $\mathbb{P}^2$, but $\dim_{\mathbb{C}}(S_d^{\Phi_n}) = 2$!), hence dense open. We get a map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}(S_d^{\Phi_n})^* \tag{22}$$

where $\mathbb{P}(S_d^{\Phi_n})^*$ is isomorphic to $\mathbb{P}^1$.

[3]ie. no two points coincide, no three are collinear, etc.

2.2.1 Blowing up to resolve indeterminacy

The map defined above is clearly not a morphism, but it can be extended to a morphism outside Φ_n, and possibly at some points of Φ_n. Note that to do this we may have to reduce the degree d to ensure the image in (18) has a basis of *coprime* elements.

If we blow up $\mathbb{P}^2$ at the points of Φ_n, then we get a morphism since we have blown $\mathbb{P}^2$ up at a set containing the indeterminacy locus of the rational map (see [11] , lecture 7). As we will see below, we may not need to blow up so much.

For example, take $d = n = 1$, then $\dim_{\mathbb{C}}(S_1^1) = 2$. Then the map is the projection to the exceptional fibre of the blowup; to be explicit, suppose $\Phi_n = \{(0:0:1)\}$. Then blow up, giving

$$\widetilde{X} : (x.x_1 = y.y_1) \subset \mathbb{P}^1_{(x_1:y_1)} \times \mathbb{P}^2_{(x:y:z)}, \tag{23}$$

and the map to $\mathbb{P}^1$ is given by

$$((x_1 : y_1), (x : y : z)) \mapsto (x_1 : y_1) \in \mathbb{P}^1 \tag{24}$$

which is clearly a morphism. It is easy to see that we needed to blow up at all of Φ_n. In our coordinate free formulation, we can view the blowup as the *closure* of the image of $\mathbb{P}^2$ sitting inside $\mathbb{P}(S_d)^*$.

For an example of when we do not need to blow up all of Φ_n, we look at the case of the cubic surface. This can be seen as $\mathbb{P}^2$ blown up at 6 points in general position. The dimension of $S_3^{\Phi_6}$ is 4, hence the blown up $\mathbb{P}^2$ embeds as the cubic surface in $\mathbb{P}^3$. It is well known that the cubic surface contains 27 lines - see [9]. Then the map projecting away from one of these lines can be extended to a morphism to $\mathbb{P}^1$ - again, see [9] for details.

2.2.2 Definitions and admissibility

Next, we want to calculate the Kodira dimension of the fibration $\widetilde{f} : \widetilde{X} \to \mathbb{P}^1$ (where $\widetilde{f}$ is defined by composition of f with π). We recall that:

$$\kappa(\tilde{f}) = \kappa(\mathbb{P}^1, K_{\mathbb{P}^1} + \Delta(\tilde{f})) \tag{25}$$

Note that this expression is valid because any fibration to a curve is admissible as shown above.

So we need to calculate $\Delta(\tilde{f})$.

Note that if $\Delta(\tilde{f}) = 0$, then $\kappa(\tilde{f}) = \kappa(\mathbb{P}^1, -2.pt) = -\infty$.

Before proceding further, we recall the definition of the multiplicity divisor of a fibration in the case where the target is a curve:

Definition 5 (Multiplicity divisor on a curve C) *Given a point $p \in C$, we say $m_p = 0$ if the fibre over p is smooth. Otherwise, let $\{\Delta_j | j \in J\}$ be the set of irreducible components of the fibre over p.*

Let $m_{p,j}$ denote the scheme theoretic multiplicity of Δ_j.

Then let $m_p := \inf(m_{p,j} | j \in J)$.

Then let $\Delta(\tilde{f}) := \sum_{p \in C}(1 - 1/m_p).p$.

2.2.3 The singular fibres

Now we want to find the singular fibres with multiplicity greater than 1. We will use an approach based on considering the fibration as an embedding of $\mathbb{P}^1$ in the Chow scheme of X, as in Section 1.7.4 , to see that in general such fibres do not exist. In the next section, we will show what the maximum possible $\Delta(f)$ is using a rather different approach (which does not generalise as well to more complicated cases).

The key observation to make is that **a fibre has multiplicity greater than 1 if and only if the derivative of the equation defining it vanishes everywhere on the fibre**. Conveniently, it is enough to show the derivative vanishes on a dense open subset of the fibre, since the set of singular points is closed. Hence we do not need to worry about the finite number of points at which we blew $\mathbb{P}^2$ up, and it suffices to only look at one affine piece of $\mathbb{P}^2$.

Write the fibration as

$$\begin{aligned} \mathbb{P}^2 &\to \mathbb{P}^1 \\ p &\mapsto (A(p) : B(p)). \end{aligned} \tag{26}$$

whereA, B are coprime forms of degree d in the coordinates $(X : Y : Z)$ on $\mathbb{P}^2$. Write $(s : t)$ for the coordinates on $\mathbb{P}^1$. Then the fibre over the point $(s : t)$ in $\mathbb{P}^2$ is defined by the equation

$$F := B.s - A.t = 0. \tag{27}$$

Working on the affine piece $Z \neq 0$ (which suffices as stated above), we see that the fibre has multiplicity greater than 1 if and only if

$$F(p) = 0 \Rightarrow \left(\frac{\partial F}{\partial X}(p) = \frac{\partial F}{\partial y}(p) = 0 \right) \tag{28}$$

ie if and only if

$$\sqrt{F} \Big| \frac{\partial F}{\partial X} and \sqrt{F} \Big| \frac{\partial F}{\partial y} \tag{29}$$

where $\sqrt{F}$ denotes a generator of the radical ideal $\sqrt{(F)}$.

Now

$$\deg \frac{\partial F}{\partial X} < \deg F, \tag{30}$$

and so we must have $\sqrt{F} \neq F$.

For example in the degree 2 case, we have a map

$$\begin{aligned} \mathbb{P}^1 &\hookrightarrow \mathbb{P}(S_2) \cong \mathbb{P}^5 \\ (s : t) &\mapsto s.A + t.B \end{aligned} \tag{31}$$

where A, B are the chosen basis of $S_d^{\Phi_n}$ (and hence independant), and the map embeds $\mathbb{P}^1$ as a *linear* subspace. We need to count points of intersection of the image of this map with the image of

$$\psi : \mathbb{P}^2 \cong \mathbb{P}(S_1) \to \mathbb{P}(S_2) \cong \mathbb{P}^5 \tag{32}$$

$$C \mapsto C^2 \tag{33}$$

where the image has degree 2 (as can be seen by considering its defining equations).

It is clear that in general the images will not meet, so that in general $\Delta(f) = 0$. However, this does not help us to determine the *maximum* possible $\Delta(f)$ or $\kappa(f)$. Considering the degree of ψ above, we see that it can meet the linearly embedded $\mathbb{P}^1$ at at most two points, so recalling the definition

$$\Delta(f) := \sum_{p \in C} (1 - 1/m_p).p, \tag{34}$$

we see that

$$\Delta(f) \leq 1.[point] \tag{35}$$

and hence that

$$\kappa(f) = \kappa(\mathbb{P}^1, \Delta(f) - 2.[point]) \leq \kappa(\mathbb{P}^1, -1.[point]) = -\infty. \tag{36}$$

What about the higher degree cases? Well there may be many more possible multiplicities for the fibres, so we have to replace ψ by several maps, some from higher dimensional projective spaces. In addition, each of these maps is likely to have degree greater than 2, so the linearly embedded $\mathbb{P}^1$ may meet a given image at more than 2 points. If either of these occured, it would certainly be possible to have $\kappa(f) \geq 0$. However, as we shall see, this is in fact impossible.

2.3 Maximising $\Delta(f)$

In this section we show that in fact the Kodiara dimension of the fibrations we are considering (of any degree) is always negative, i.e. that $\Delta(f) < 2.[point]$, or equivalently that there can be a most two points over which the fibre is multiple. We also give an upper bound on $\Delta(f)$, and show that it is attained. The proof of the following proposition is origional to the best of my knowledge.

Proposition 5 *Fix an integer $d \geq 2$. Let $A, B \in S_d$ coprime. Let*

$$\begin{aligned} f : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ p &\mapsto (A(p) : B(p)) \end{aligned}$$

Then there are at most 2 points in $\mathbb{P}^1$ which have fibres of multiplicity greater than 1.

Proof:

Suppose there are three distinct points $(s_i : t_i) \in \mathbb{P}^1, (i = 1, 2, 3)$ which have fibres $\mathbb{P}^2_{(s_i:t_i)}$ of multiplicity greater than 1.

This corresponds to the existance of three distinct collinear points in the locus of multiple fibres inside $\mathbb{P}(S_d)$.

Call the points p_1, p_2, p_3.

Then let the fibres over p_i be:

$$f^{-1}(p_i) : (U_i^{d_i}.V_i = 0)$$

where the U_i are reduced homogeneous forms in x, y, z, and $U_i \nmid V_i$.

Let $f := \max_i(\deg(U_i), \deg(V_i))$.

Let ${}^4S_f$ denote the space of forms of homogeneous degree f in 4 variables x, y, z, t.

Let $W_1, \ldots, W_k$ be a basis for the orthogonal complement of $\langle U_1, U_2, V_1, V_2\rangle \subset^4 S_f$ (where we multiply through by the fourth variable t to make the U_i, V_i all have the same homogeneous degree f.

Then we can write any degree f form in x, y, z as a linear combination of U, V, W, so write

$$U_3 = \alpha_1 U_i + \alpha_2 U_2 + \sum_{i=1}^{k} \beta_i W_i + \alpha_3 V_1 + \alpha_4 V_2$$

and

$$V_3 = \alpha'_1 U_i + \alpha'_2 U_2 + \sum_{i=1}^{k} \beta'_i W_i + \alpha'_3 V_1 + \alpha'_4 V_2$$

Then by our assumption of collinearity, we can write

$$U_3^{d_3} V_3 = a U_1^{d_1} + b U_2^{d_2} V_2$$

for some $a, b \in \mathbb{C}$.

Thus $\beta_i = \beta'_i = 0$ for every i. Similarly, by considering the cross terms in the binomial expansion of $U_3^{d_3} V_3$, and using the fact that $U_i \nmid V_i$ and U_1, U_2 etc are coprime, we get that $U_3 = V_3 = 0$, and hence a contradiction. $\square$

With the result above we can easily see $\kappa(f) = -\infty$:

$$\Delta(f) := \sum_{p \in C} (1 - 1/m_p).p, \tag{37}$$

and $m_p \geq 2$ so

$$\Delta(f) < 2.[point] \tag{38}$$

and hence

$$\kappa(f) = \kappa(\mathbb{P}^1, \Delta(f) - 2.[point]) < \kappa(\mathbb{P}^1, 0.[point]) = 0. \tag{39}$$

2.3.1 Maximising $\Delta(f)$

Further, we can draw a straight line between any two points in $\mathbb{P}^N$, $N \geq 1$, so any pair of multiple fibres can be obtained. In particular, the fibration

$$\begin{array}{rl} f : \mathbb{P}^2 & \to \mathbb{P}^1 \\ (X : Y : Z) & \mapsto (X^d : Y^d) \end{array}$$

has only one point of indeterminacy, at $(0 : 0 : 1)$, so for any d we get a fibration of degree d from the bowup of $\mathbb{P}^2$ at one point with $\Delta(f) = 2\frac{(d-1)}{d}.[point]$, so as $d \to \infty$, $\Delta(f)$ can get arbitrarily close to $2.[point]$, but never get there.

3 Fibrations $X \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$, $\kappa(X) \geq 0$

3.1 The Kodaira dimension of X

3.1.1 Adjunction

We consider $X = X_{m,d} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ (where $X_{m,d}$ just means X has homogeneous degree m in the variables on $\mathbb{P}^1$ and d in the variables on $\mathbb{P}^2$)

Our fibration will be the projection to the first factor, $\mathbb{P}^1$. We will denote this map f.

If we take $m = d = 1$, we get that X is the blowup of $\mathbb{P}^2$ at one point. This connects the new class of fibration to those studied in Section 2. The $X_{m,d}$ are certainly not strict generalisation of those in Section 2: for example in this section X need not always have $\kappa(X) = -\infty$.

First we will calculate the Kodaira dimension of X, for which we need to know the canonical class K_X

For this, we notice that X is a hypersurface in the smooth variety $\mathbb{P}^1 \times \mathbb{P}^2$, and so we can use the *adjunction formula* (taken from [9], Pp74). This says that if X is a hypersurface in a smooth variety Y, then:

$$K_Y = (K_X + X)|_X. \tag{40}$$

So for this we need to know K_Y, so we want to look at the divisor class group $Cl(\mathbb{P}^1 \times \mathbb{P}^2)$ (that is the group of cycles of 2 dimensional subvarieties of $\mathbb{P}^1 \times \mathbb{P}^2$ up to linear equivalence ($\mathbb{P}^1 \times \mathbb{P}^2$ is smooth, so our choice of definition is not very important)).

3.1.2 The class group and canonical divisor of $\mathbb{P}^1 \times \mathbb{P}^2$

If we put coordinates $(s : t)$ on $\mathbb{P}^1$ and $(x : y : z)$ on $\mathbb{P}^2$, then it is easy to see that:

$$Cl(\mathbb{P}^1 \times \mathbb{P}^2) \cong Z \times Z \cong Z.[s = 0] \times Z.[x = 0]. \tag{41}$$

Then we know that $K_{\mathbb{P}}^1 = -2.[s = 0]$ and $K_{\mathbb{P}}^2 = -3.[x = 0]$, so considering the sheaf of differentials on a product of varieties we see that:

$$K_{\mathbb{P}^1 \times \mathbb{P}^2} = -2.[s = 0] + -3.[x = 0]. \tag{42}$$

3.1.3 The class group and canonical divisor of X

To add this to X we need to know the equivalence class of X in $Cl(\mathbb{P}^1 \times \mathbb{P}^2)$. But $X = X_{m,d}$, so we get:

$$[X] = m.[s = 0] + d.[x = 0] \tag{43}$$

hence

$$[K_Y + X] = (m - 2).[s = 0] + (d - 3)[x = 0]. \tag{44}$$

We can move X and $[s = 0], [x = 0]$ such that both $[s = 0]$ and $[x = 0]$ meet X transversely and at nonsingular points (assuming X has only isolated singularities, which we will check by hand for the examples we will consider).

Thus K_X is given by the same cycle $(m - 2).[s = 0] + (d - 3)[x = 0]$, but this time inside X.

3.1.4 $\kappa(X)$

From the above we see that $\kappa(X) \geq 0$ if and only if $m \geq 2$ and $d \geq 3$. In fact, we can produce a table of values of $\kappa(X)$ as follows:

m:	<2	2	3	4	5	$\cdots$
$d<3$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\cdots$
$d=3$	$-\infty$	0	1	2	2	$\cdots$
$d=4$	$-\infty$	2	2	2	2	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

3.1.5 Singularities on X

X as above will not always be smooth, but we want to look at fibrations from smooth X. As long as X has only *isolated point* singularities, we can blow these up to give a smooth $\widetilde{X} \xrightarrow{\pi} X$. This may change K_X by adding in a number of copies of each exceptional fibre of the blowup. We can write:

$$K_{\widetilde{X}} = \pi^*(K_X) + r.E \tag{45}$$

(where E is an exceptional fibre). It is easy to see that this will not affect the Kodira dimension of X if $r \geq 0$; any function on X satisfying K_X will still be permitted, and no function can vanish only on the exceptional fibres. We give a condition to have $r \geq 0$:

Proposition 6 *If the exceptional locus of a blowup as above has exceptional locus E a union of copies of $\mathbb{P}^1$, then $r \geq 0$ in formula (45)*

Proof:
$\widetilde{X}$ is smooth, so we can apply adjunction to E, obtaining

$$K_E = (K_{\widetilde{X}} + E)\big|_E$$

Taking degrees on both sides, we obtain:

$$\begin{aligned} \deg(K_E) &= \deg\left((\pi^* K_X + rE + E).E\right) \\ &= (r+1)E.E \end{aligned}$$

since we can move K_X so that it does not meet the singular points. Now from [9] (Chapter A), we know that E has negative self intersection. Also, if E is a union of copies of $\mathbb{P}^1$ then $\deg(K_E) < 0$, so we must have $r \geq 0$ as required. □

Thus we will need to check in examples we calculate that the exceptional locus of and singularity is a collection of copies of $\mathbb{P}^1$.

In fact, such blowups also do not affect $\kappa(f)$, as will be explained below.

3.2 $\kappa(f)$, the Kodaira dimension of the fibration

3.2.1 The effect that blowing up the singularities of X has on $\kappa(f)$

Blowing up will never reduce the multiplicity divisor. Let $q \in X, p := f(q)$. If X is singular at q, then X_p will be singular. (However, X_p being singular does not imply that X is singular at any point over p, so the multiplicity divisor $\Delta(\widetilde{f})$ need not be zero in general!)

It is easy to see that the corresponding fibre $\widetilde{X}_p$ will be the *proper* transform of the fibre X_p under the blowup π. The fibre X_p is globally principle, so considering the blowup in local coordinates easily shows that the multiplicity of $\widetilde{X}_p$ is the same as that of X_p.

This is in contrast with the situation in Section 2 where the blowups are used to resolve indeterminacy in the fibration; in that situation, the fibre in the blowup is the *strict* transform of the non-blown up fibre, wheras here the fibration is well defined before the blowup, and the fibre that contained the singular point is the *proper* transform of the non-blown-up fibre, and hence will contain the exceptional divisor (with multiplicity divisible by the multiplicity of X_p).

3.3 Finding the singular fibres

3.3.1 What the singular fibres look like

We consider singular fibres of X. Each fibre is a curve in $\mathbb{P}^2$ of degree d (and so has arithmetic genus $(d-1)(d-2)/2$).

Take the example $d = 3$. In this case, the possibilities for a singular fibre S are:

1. a (reduced and irreducible) cubic
2. a conic union a line
3. a double line and a reduced line
4. a tripple line

Considering the definition of m_p, we see that only in the fourth case do we get $m_p \neq 0$, and that in this case $m_p = 3$.

In degree 4 or greater, things get more complicated, for example you can have the union of two double lines, or a double conic.

We want to find how many and what multiplicity of singular fibres there are for a given fibration $X_{m,d}$.

Fix m , d . Writing $X : \left(\sum_{i+j+k=3} a_{i,j,k} x^i y^j z^k\right)$ where each $a_{i,j,k}$ has degree m, we can any such such fibration by a map:

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^{N-1} \tag{46}$$

where $N = \binom{d+2}{2}$, and φ sends $(s:t)$ in $\mathbb{P}^1$ to one of the $a_{i,j,k}((s:t))$ in each factor of $\mathbb{P}^{N-1}$.

Clearly the space of all such fibrations equals the space of all such degree m maps to $\mathbb{P}^{N-1}$.

As $\mathbb{P}^{N-1}$ parametrises the fibres, it makes sense to ask how the singular fibres are distributed in $\mathbb{P}^{N-1}$.

3.3.2 The singular fibres for $d = 3$

For $d = 3$, the only singular fibres with multiplicity greater than 1 are tripple lines. These correspond to the image of the map:

$$\psi : \mathbb{P}^2 \to \mathbb{P}^9 \tag{47}$$

$$(a : b : c) \to (a^3 : b^3 : c^3 : 3a^b : 3a^2c : \cdots : 6abc) \tag{48}$$

From this, it is clear that in general the images of ψ and φ will not meet, so there are no tripple fibres in general. We can also see that, as φ has degree m and ψ has degree 3, so they may meet at at most $3m$ distinct points.

Suppose this was attained. Then

$$\kappa(f) = \kappa(\mathbb{P}^1, (3m(1 - 1/3) - 2).point) = \kappa(\mathbb{P}^1, (2m - 2).point), \tag{49}$$

so

$$\kappa(f) = \begin{cases} -\infty & \text{if } m < 1 \\ 0 & \text{if } m = 1 \\ 1 & \text{if } m > 1 \end{cases} \tag{50}$$

We note however that this need not ever occur; for example we have already shown that if $m = 1$, $\kappa(f) = -\infty$.

To generalise this approach for all d, we need to look more carefully at what the relevant singular fibres are and how they can occur.

3.3.3 The singular fibres for general d

For $d = 2$, this is pairs of lines.

For $d = 3$, tripple lines.

For $d = 4$, we can have a double conic, or pair of double lines, or a quadruple line [4]. However, a quadruple line is just a special case of a pair of double lines, which is itself just a special case of a double conic. Note that we are looking for the maximum value of the Kodiara dimension, and so as the special cases either do not change $\Delta(f)$, or reduce it, we can ignore them for now, though we will have to be aware of them when seeking examples. As such, it suffices to parametrise all double conics.

For $d = 5$, the only relevant case is that of a double line plus a tripple line, resulting in multiplicity 2. [5]

For $d = 6$, there are two substantially different possibilities; a double cubic, or a tripple conic. However, we are looking to maximise $\Delta(f)$, so clearly the best case is that of the tripple conic.

3.3.4 Evaluating Ψ_d

Let $\Psi = \Psi_d$ denote the number of maps $\psi : \mathbb{P}^2 \to \mathbb{P}^{N-1}$ which are needed to parametrise all the fibres of multiplicity > 1. We have just seen that

$$\Psi_d = \begin{cases} 1 & \text{if } 1 \leq d \leq 5 \\ 2 & \text{if } d = 6 \end{cases} \tag{51}$$

[4] Note that a tripple line and a single line has multiplicity 1 due to the inf in the definition

[5] under the older definition of multiplicity using gcd instead of inf, the multiplicity would have been 1 in this case, and we would have had to look at quintuple lines to get $\Delta(f) \neq 0$.

and it is easy to see that $\Psi_d \geq 1$ for $d > 6$.

Now we have that [6]

$$\kappa(f) \leq \kappa(\mathbb{P}^1, (m.d.\Psi_d.(1-1/3)-2).point) \tag{52}$$

and hence for $1 \leq d \leq 5$,

$$m.d.\Psi_d.(1-1/3)-2 \leq 3.m.d/2-2. \tag{53}$$

and for $d \geq 6$,

$$3.m.d/2-2 \geq m.d.\Psi_d.(1-1/3)-2. \tag{54}$$

3.4 Comparing $\kappa(f)$ and $\kappa(X)$

From these we can easily calculate an upper bound on the Kodiara dimension of any fibration of the type we have been considering:

m:	1		2		3		4		5	
$d=1$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$d=2$	$-\infty$	$-\infty$	0	$-\infty$	1	$-\infty$	1	$-\infty$	1	$-\infty$
$d=3$	0	$-\infty$	1	0	1	1	1	2	1	2
$d=4$	0	$-\infty$	1	2	1	2	1	2	1	2
$d=5$	1	$-\infty$	1	2	1	2	1	2	1	2
$d=6$	1	$-\infty$	1	2	1	2	1	2	1	2

where for each value of m, d the left hand box gives the upper bound we have calculated on $\kappa(f)$, and the right hand box gives the value of $\kappa(X)$. Some correlation is apparent between these, especially when we recall that $\kappa(f) \leq 1$ and $\kappa(X) \leq 2$ because of the dimensions of $\mathbb{P}^1$ and X respectively.

For $m, d > 6$, the resulting upper bounds continue the obvious pattern.

3.5 Achieving the upper bound I

The next question is whether these upper bounds are attainable.

The first main barrier is that if X has non-isolated singularities, or isolated singularities with complicated exceptional fibres, it is possible that the resolution will change the degree of the general fibre.

The second is that in general in (52), it is not clear whether the term $m.d$ is achievable.

In fact, we do not need to know about either of these; we will construct fibrations to show that in most cases our upper bounds are achievable, and it is easy to check that for the examples we construct the general fibres have the desired properties.

We will show below that for $d \geq 4$ and $m \geq 2$, these upper bounds are effective.

The case $m = 1$ is a special case of what we considered in Section 2, and so the same upper bounds on $\kappa(f)$ and $\Delta(f)$ apply, namely that $\kappa(f) = -\infty$. This is considerably stronger than the result we obtained previously, so rendering that redundant. However, the methods we used in Section 2 to obtain these bounds do not generalise easily to the case where $d \geq 2$.

[6] see definition 4 for the definition of Kodaira dimension κ for $\mathbb{Q}$-divisors.

3.6 An example of an elliptic fibration

Our first example has $m = 1$ and $d = 3$. We will take a fibration where all the fibres are in Weierstrass form for simplicity; this imposes very strong conditions on multiple fibres, making them easy to find. We choose our defining equation

$$F := sZY^2 - sX^3 + sZX^2 + tZ^3 \tag{55}$$

It is easy to see that for a cubic in Weierstrass form, the only way for it to have multiplicity greater than 1 is for it to be defined by $X^3 = 0$ or $Z^3 = 0$. So the only multiple fibre for this fibration occurs at $(s : t) = (0 : 1)$. X is singular at the point $(0 : 1)(0 : 1 : 0)$, with exceptional locus $\mathbb{P}^1$, so as shown in 3.2.1, this will not affect the multiplicity of the fibre. Hence

$$\Delta(f) = (1 - 1/3).[point] = 2/3.[point]. \tag{56}$$

However,

$$\kappa(f) = -\infty \tag{57}$$

since $K_{\mathbb{P}^1} = -2.[point]$.

For another example, take

$$F := s^3Y^2Z - t^3X^3 - S^3Z^2X - T^3Z^3 \tag{58}$$

Then there are no multiple fibres, so again $\Delta(f) = 0$. A (somewhat laborious) check by hand shows there are only isolated singularities. Without checking what the exceptional fibres are, we cannot be sure what $\kappa(X)$ is.

So far we have yet to see an example with $\Delta(f) \neq 0$, but with the aid of the theory that has been developed above, we can demonstrate a family of such fibrations. We have reduced the problem to one of counting intersections of varieties in $\mathbb{P}^N$.

3.7 Attaining the upper bound II

We have shown that for $m \geq 3, d \geq 4$ the upper bound for $\kappa(f)$ is 1. We now give a family of examples showing that these upper bounds can be attained.

We follow the approach used in Section 3.3 to produce the upper bound. It is clear that to maximise $\kappa(f)$, we need to give a curve of degree m meeting the singular fibres in the Chow scheme of X in as many points as possible. In the examples we give, the fibres will all have multiplicity d, so to maximise $\kappa(f)$ it is enough to find 3 singular fibres. [7]

Define $X \subset \mathbb{P}^2_{(X:Y:Z)} \times \mathbb{P}^1_{(s:t)}$ by:

$$F(X, Y, Z, s, t) := \frac{st}{2}(s^{m-2} + t^{m-2})((X + Y + Z)^d - Y^d - Z^d) + s^mY^d + t^mZ^d$$

It is clear that X has tripple lines for fibres over the points $(s : t) = (0 : 1), (1 : 0)$ and $(1 : 1)$. It remains to check that all the singularities on X are isolated, and the exceptional loci are collections of $\mathbb{P}^1$s (so we can apply the result of Section 3.2.1).

[7] It is not clear what the maximum value for $\Delta(f)$ will be, so we are only working to maximise $\kappa(f)$.

Several pages of computation (see the appendix for details) show that for $m \geq 3, d \geq 4$,:

• the fibre over $(0:1)$ is singular at the point defined by $Z = 0, (X+Y)^d = Y^d$. The exceptional locus is a bunch of $\mathbb{P}^1$s.

• the fibre over $(1:0)$ is similar to the one above by symmetry, with isomorphic exceptional locus.

• the fibre over $(1:1)$ is singular only at the points defined by $X+Y+Z = 0, Y^d = Z^d$, with exceptional locus a bunch of $\mathbb{P}^1$s again.

• X is smooth outside these fibres.

Thus the corresponding fibration from the blowup $\widetilde{X}$ (of Kodaira dimension 2) of X at these points to $\mathbb{P}^1$ has Kodaira dimension 1, so our upper bounds are effective for $m \geq 3, d \geq 4$.

4 Further lemmas on $\kappa(f)$

In this section we give some more lemmas on $\kappa(f)$, partly for interest and as motivation for Section 5, and partly to give a better background to Section 5; the results of Section 6 do not in fact use the results of this section, but the methods used are in some ways more abstract generalisations of the methods in this section.

4.0.1 $\kappa(f)$ is invariant under modifications of the top space

Let $f : X \to Y$ and $f' : X' \to Y'$ be two equivalent holomorphic fibrations between smooth spaces, so we have bimeromorphic maps $u : X \to X'$ and $v : Y \to Y'$ such that

$$\begin{array}{cccc} & X & \stackrel{u}{\to} & X' & \\ f & \downarrow & \Box & \downarrow & f' \\ & Y & \stackrel{v}{\to} & Y' & \end{array} \tag{59}$$

commutes. Then we have [in a similar vein to [3]]:

Proposition 7 *Suppose $Y = Y'$ and v is identity. Suppose also that u is holomorphic. Then*

$$\Delta(f \circ u) = \Delta(f) \tag{60}$$

and hence

$$\kappa(Y/\Delta(f \circ u)) = \kappa(Y/\Delta(f)). \tag{61}$$

Proof:

Following the same notation as Section 1.4, let V be a codimension 1 subvariety of Y. We wish to show that the multiplicity m_V is the same for $f' = f \circ u$ as for f. We write

$$f^*(V) = \sum_E m_E \cdot E + R$$

and

$$f'^*(V) = \sum_D m_D \cdot D + T = (f \circ u)^*(V) = u^* \circ f^*(V)$$

$$= u^*(\sum_E m_E \cdot E + R) = \sum_E m_E \cdot u^*(E) + u^*(R)$$

where all sums run over all the codimension 1 subvarieties of X' which map surjectively to V, and R and T are exceptional (recalling that V is irreducible).

$u^*(R)$ is clearly $f \circ u$ exceptional; R is f-exceptional, and composing with u cannot reverse the drop in codimension. Thus $u^*(R) \leq T$, since T is the $f \circ u$ exceptional part of $f'^*(V)$. Hence

$$\sum_D m_D \cdot D \leq \sum_E m_E \cdot u^*(E).$$

The reverse inclusion need not hold, so we need to consider the u-exceptional part of $u^*(D)$.

For each E, we can write $u^*(E) = \bar{E} + R_E$ where $\bar{E}$ is the strict transform of E by u and R_E is its u-exceptional part. Then we get

$$\sum_D m_D \cdot D \leq \sum_E m_E \cdot (\bar{E} + R_E)$$

and also

$$\sum_D m_D \cdot D \geq \sum_E m_E \cdot \bar{E}$$

since although the R_E need not be $f \circ u$-exceptional in general, they will certainly contain all the $f \circ u$-exceptional parts of T.

Then by considering the multiplicity of any (reduced and irreducible) component, we see that the infinitum of the multiplicities will be the same.

□

4.0.2 How $\kappa(f)$ changes under modifications of the base

Having studied how we can change the top space without affecting the Kodiara dimension, we show the nearest we can get for modifications of the base. Note that the inequality we will produce may be strict, as an example of Campana's shows. In Section 5, we will explain Campana's example, and investigate why his method of construction is unlikely to give an example of inequality when $\Delta(f)$ is a normal crossing divisor.

We have [in a similar vein to [3]]

Proposition 8 *Let $f : X \to Y$ and $f' : X' \to Y'$ be two equivalent holomorphic fibrations, so we have bimeromorphic maps $u : X \to X'$ and $v; Y \to Y'$ such that*

$$\begin{array}{ccccc} & X' & \overset{u}{\to} & X & \\ f' & \downarrow & \square & \downarrow & f \\ & Y' & \overset{v}{\to} & Y & \end{array} \qquad (62)$$

commutes.

Suppose in addition that u and v are holomorphic. Then

$$\Delta(f') = v^*(\Delta(f)) + R$$

for some $\mathbb{Q}$-divisor R supported on the exceptional locus of v, and hence

$$v_*(\Delta(f)) = \Delta(f)$$

and

$$\kappa(Y'/\Delta(f')) \leq \kappa(Y/\Delta(f)).$$

Proof:

By Proposition 7, we can and shall assume $X' = X$.

Following the same notation as Section 1.4, let V be a codimension 1 subvariety of Y'. Since our varieties are compact, $v(V)$ is a variety. Then we have

$$f^*(v(V)) = (v \circ f')^*(v(V)) = f'^* \circ v^*(v(V)). \tag{63}$$

$v(V)$ may have codimension 2 or more in Y, in which case m_V is undefined. But in that case the contribution of V to $\Delta(f')$ is obviously supported on the exceptional locus of v, so that is fine.

Thus we can assume $v(V)$ has codimension 1 in Y. We then write

$$v^*(v(V)) = V' + E$$

with E an effective v-exceptional divisor on Y'. Then writing

$$f^*(V) = \sum_D m_D \cdot D + T$$

where all sums run over all the codimension 1 subvarieties of X' which map surjectively to V, we get by (63)

$$\sum_D m_D \cdot D + T = (f')^*(V') + (f')^*(E).$$

Now $(f')^*(E)$ is clearly f-exceptional, and T contains all the f-exceptional part of $f^*(V)$, so $f^*(V) \leq T$, and hence

$$\sum_D m_D \cdot D = (f')^*(V') + ((f')^*(E) - T)$$

where $(f')^*(E) - T$ is clearly v exceptional, so we have the first part.

Next we want to show

$$v_*(\Delta(f)) = \Delta(f)$$

and

$$\kappa(Y'/\Delta(f')) \leq \kappa(Y/\Delta(f)).$$

For the first one, note that R is v-exceptional, and so $v_*(R) = 0$, so by applying v_* to both sides we are done.

Writing $R = R^+ - R^-$ with R^+ and R^- effective and v-exceptional, we get

$$\kappa(Y'/\Delta(f')) \leq \kappa(Y', K_{Y'} + v^*(\Delta(f)) + R^+) = \kappa(Y'/v^*(\Delta(f)))$$

where the equality holds because a global section cannot vanish only on an exceptional divisor.

Then

$$\kappa(Y'/v^*(\Delta(f))) = \kappa(Y/\Delta(f))$$

by considering composing global sections with v.

□

4.0.3 Another admissibility result

Note that the following result gives another proof of admissibility for smooth curves of genus greater than 0, but as $\mathbb{P}^1$ is the case in which we are particularly interested, this is not particularly relevant.

Proposition 9 (Admissibility when $\kappa(Y) \geq 0$) *Let $f : X \to Y$ and $f' : X' \to Y'$ be two equivalent smooth holomorphic fibrations, with $\kappa(Y) \geq 0$. Suppose that f' dominates f.*

Then

$$\kappa(f) = \kappa(Y'/\Delta(f')) = \kappa(Y/\Delta(f))$$

Proof:

We have holomorphic bimeromorphic maps $u : X \to X'$ and $v; Y \to Y'$ such that

$$\begin{array}{ccccc} & X' & \overset{u}{\to} & X & \\ f' & \downarrow & \square & \downarrow & f \\ & Y' & \overset{v}{\to} & Y & \end{array} \tag{64}$$

commutes (recalling X, Y, X', Y' compact smooth irreducible complex spaces).

From Proposition 8, we have $\kappa(Y'/\Delta(f')) \leq \kappa(Y/\Delta(f))$. If we can show equality here, then we are done by definition of the Kodaira diension - it takes the infimum of the Kodaira dimensions over all models.

So it suffices to show $\kappa(Y'/\Delta(f')) \geq \kappa(Y/\Delta(f))$.

So it suffices to show $K_{Y'} + \Delta(f') \geq v^*(K_Y + \Delta(f))$.

Let E denote the exceptional locus of v, and let $\overline{\Delta(f)}$ denote the strict transform of $\Delta(f)$ by v.

Now Y is smooth, so we get:

- For some b in the non-negative rationals,

$$v^*(\Delta(f)) - b \cdot E \leq \overline{\delta(f)}. \tag{65}$$

Well clearly this holds outside E, so it holds everywhere by subtracting off enough copies of E.

-

$$\Delta\bar{(}f) \leq \Delta(f') \tag{66}$$

Again, this is clear outside E - the diagram above commutes, and Proposition 3 means that we can assume $X = X'$ and that v^{-1} is well defined outside E. But again, on E, $\overline{\Delta(f)}$ is the 'smallest' it could be such that $\overline{\Delta(f)}$ is a divisor (ie a sum of codimension 1 subvarieties).

- For some a in the non-negative rationals,

$$K_{Y'} \geq v^*(K_Y) + a \cdot E. \tag{67}$$

This again follows from Proposition 3.

We can combine these three to get

$$\begin{aligned} K_{Y'} + \delta(f') \quad \geq v^*(K_Y) \quad & + a \cdot E + v^*(\Delta(f)) - b \cdot E \\ & = v^*(K_Y + \Delta(f)) + (a - b) \cdot E. \end{aligned} \tag{68}$$

So we are done if $(a - b) \geq 0$.

So assume not, i.e. that $a/b < 1$.

Then by (65),

$$b \cdot E \geq v^*(\Delta(f)) - \overline{\Delta(f)})$$

so

$$a \cdot E = (a/b) \cdot b \cdot E \geq (a/b) \left[v^*(\Delta(f)) - \overline{\Delta(f)} \right]$$

so

$$\overline{\Delta(f)} + a \cdot E \geq v^*(\Delta(f)) + (1 - a/b)\overline{\Delta(f)} \tag{69}$$

so

$$\begin{aligned} K_{Y'} + \delta(f') &\geq v^*(K_Y) + \overline{\Delta(f)} + a \cdot E && \textit{(by (68) and (65))} \\ &\geq (a/b)v^*(K_Y + \Delta)f)) + (1 - a/b) \left[v^*(K_Y) + \overline{\Delta(f)} \right] && \textit{(by (69))} \\ &\geq (a/b)v^*(K_Y + \Delta)f)) \end{aligned}$$

where the last inequality holds because $\kappa(Y) \geq 0$;

we need to show $(1 - a/b) \left[v^*(K_Y) + \overline{\Delta(f)} \right] \geq 0$.

Well by assumption $1 - a/b > 0$, and $\overline{\Delta(f)} \geq 0$ by definition. It suffices to show $K_Y \geq 0$, but this is implied by $\kappa(Y) \geq 0$, so we have the inequality.

Then

$$K_{Y'} + \delta(f') \geq (a/b)v^*(K_Y + \Delta)f))$$

so we get

$$\begin{aligned} \kappa(Y'/\delta(f')) &= \kappa(Y', k_{Y'} + \Delta(f')) \\ &\geq \kappa(Y', (a/b)v^*(K_Y + \Delta(f)) \quad = \kappa(Y', v^*(K_Y + \Delta(f)) \\ &\qquad = \kappa(Y, K_Y + \Delta(f)) \quad = \kappa(Y/\Delta(f)) \end{aligned}$$

where the equality on the middle line holds by definition of the Kodaira dimension of a $\mathbb{Q}$-divisor, and the fact that $a/b < 1$. □

5 Preparedness in Proposition 8

In Proposition 8, we showed that given two equivalent fibrations f, f' as in

$$\begin{array}{ccccc} & X' & \overset{u}{\rightarrow} & X & \\ f' & \downarrow & \square & \downarrow & f \\ & Y' & \overset{v}{\rightarrow} & Y & \end{array} \tag{70}$$

such that u and v are holomorphic, then

$$\kappa(Y'/\Delta(f')) \leq \kappa(Y/\Delta(f)).$$

In this section we reproduce an example from [3] showing that this inequality can be strict. We then investigate obstructions to using the same approach to find an example of strict inequality *when $\Delta(f)$ is a normal crossing divisor on a smooth Y.*

Campana's example (from [3]):

Let $Y = \mathbb{P}^2$ and Δ_{red} :=the union of $2k \geq 6$ distinct lines meeting at $a \in \mathbb{P}^2$.

Let $p : Y^+ \to Y$ be the double cover branched exactly along Δ_{red} (for example, let $\phi : Y \to \mathbb{P}^1$ vanishing exactly along Δ_{red}, then let $Y_+ : ((f-x)^2 = 0) \subset Y \times \mathbb{P}^1_{(x)})$.

Let $h : Y^+ \to Y^+$ be the map interchanging the sheets of Y^+.

Let C be an elliptic curve, and $t : C \to C$ a translation of order 2.

Let $X^+ := C \times Y^+$, $j := t \times h$.

Let $X_0 := X^+ / \langle j \rangle$.

Then we have maps

$$F_0 : \begin{array}{ll} X_0 & \to Y \\ (c, y) & \mapsto p(y) \end{array}$$

and

$$F^+ : \begin{array}{ll} X^+ & \to Y^+ \\ (c, y) & \mapsto p(y) \end{array}$$

Let $d : X \to X_0$ be a desingularisation, induced by a desingularisation of Y^+, and let $f := d \circ F_0 : X \to Y$.

Having set this up, we calculate $\kappa(Y/\Delta(f))$:

It is easy to see $\Delta(f) = \frac{1}{2}\Delta_{red}$.

Letting H denote the class of a line in $\mathbb{P}^2$, we get:

$$\begin{aligned} \kappa(Y/\Delta(f)) &= \kappa(Y, K_Y + \Delta(f)) \\ &= \kappa(\mathbb{P}^2, -3H + \tfrac{2k}{2}H) \\ &= \kappa(\mathbb{P}^2, (k-3)H) = \begin{cases} -\infty & \text{if } k \leq 2 \\ 0 & \text{if } k = 3 \\ 2 & \text{if } k \geq 4 \end{cases} \end{aligned}$$

Now we want to construct the fibration 'f'' .

Let $v : Y' \to Y$ be the blowup at a. Let F be the exceptional locus of the blowup $= v^{-1}(a)$.

We now need to construct a modified X' which will give us 'the same' fibration outside the exceptional locus F, and be well defined over F. To do this, we let Y'^+ be the double cover of Y' ramified along $\overline{\Delta_{red}}$, the strict transform of Δ_{red} be v. We let h' be the map interchanging the sheets of Y'^+, and t a translation of order 2 on an elliptic curve E as before. Then

$$X' := \frac{Y'^+ \times E}{\langle h' \times t \rangle}$$

with the map $X' \to Y'$ the projection as before.

Then:

Proposition 10 $\Delta(f') = \overline{\Delta(f)}$ *(the strict transform of* $\Delta(f)$ *by* v*).*

Proof:

$$\Delta(f') = \sum_{D\subset Y'\text{irreducible divisor}} \left(1 - \frac{1}{m_D}\right) D$$

If D does not contain F, this is clearly what we want, because the blowup is an isomorphism outside the excptional locus. So it suffices to consider $D = F$.

Consider the fibre over $F \cong \mathbb{P}^1$; it is given by

$$\frac{\mathbb{P}^1 \times E}{\langle h' \times t|_{\mathbb{P}^1}\rangle}$$

h' has non-zero stabiliser only at finitely many $(2k)$ points of F. Thus over all but those $2k$ points the fibre is smooth. As such, F is not contained in the set of points over which the fibre is not smooth, so it does not appear in $\Delta(f')$. $\square$.

So noting $\overline{H} = v^*(H) - F$, we replace $v^*(H)$ by H (for convenience, so now H denotes the class of a hyperplane in the blowup Y') and write:

$$\begin{aligned} \kappa(Y'/\Delta(f')) &= \kappa(Y', -3H + F + \overline{\Delta(f)}) \\ &= \kappa(Y', -3H + F + k(H - F)) \\ &\quad \kappa(Y', (k-3)H + (1-k)F) \\ &\quad \kappa(Y', (k-3)(H - F) - 2F) \end{aligned}$$

so $\kappa(Y'/\Delta(f')) = -\infty$ exactly when $k - 3 < 0$ *or* $k - 3 - k + 1 < 0$. So $\kappa(Y'/\Delta(f')) = -\infty$ for every $k \geq 1$.

So if $k \geq 3$, we get $\kappa(Y'/\Delta(f')) < \kappa(Y/\Delta(f))$.

We now prove the above construction cannot yield a strict inequality with $\Delta(f)$ having normal crossing support on a smooth surface Y with *free abelian finitely generated class group.*

Recall the definition of a normal crossing divisor on a smooth variety Y: it is a reduced divisor D such that every component of D is smooth, and which is given (in an analytic neighbourhood of any point of D) by $x_1.x_2.\ldots.x_k = 0$ where $k \leq \dim Y$ (up to an analytic change of coordinates).

A divisor has *normal crossing support* if the corresponding reduced divisor is a normal crossing divisor.

Let Y be a smooth surface with free abelian finitely generated class group, so $Cl(Y) = \langle H_1, \ldots H_s\rangle \cong \mathbb{Z}^s$. Let the canonical divisor class K_Y of Y be given by $K_Y = \sum_i \alpha_i H_i$.

Let Δ_{red} be a normal crossing divisor on Y. Write $D = \sum_i r_i D_i$, and let ϕ be any subset of the points where Δ_{red} is not smooth (ie where the irreducible components meet). Let F be the exceptional locus of the blowup of Y at all the points of ϕ.

Let $f : Y^+ \to Y$ be the n-fold cover of Y branched exactly along Δ_{red}.

As before, construct a fibration $f : X \to Y$ of smooth spaces. Clearly $\Delta(f) = \frac{1}{n}\Delta_{red}$.

Then

$$\kappa(Y/\Delta(f)) \quad = \kappa(Y, \textstyle\sum_i (r_i - \alpha_i)H_i)$$

$$\begin{cases} -\infty & \text{if} \quad \forall i, r_i < \alpha_i \\ 0 & \text{if} \quad \forall i, r_i = \alpha_i \\ 1 & \text{if} \quad (\exists j) s.t. [r_j \leq \alpha_j \text{ and } (\forall i \neq j), r_i > \alpha_i] \\ 2 & \text{if} \quad \exists \text{ at least two } i \text{ s.t. } r_i > \alpha_i \end{cases}$$

Now we blow up Y at all points of ϕ to give Y', and then modify X to X' as above to give a fibration $f' : X' \to Y'$. We calculate $\kappa(Y'/\Delta(f'))$. Again $\Delta(f') = \overline{\Delta(f)}$ (by the same argument as above), so we get:

$$K_{Y'} + \Delta(f') \quad = \underbrace{-\sum_i \alpha_i v^*(H_i) + F}_{K_{Y'}} + \underbrace{\sum_i \frac{r_i}{n} v^*(H_i) - \frac{2}{n}F}_{\Delta(f')}$$

$$= \tfrac{1}{n}\left(\textstyle\sum_i (r_i - n\alpha_i) v^*(H_i) + (n-2)F\right)$$

So $\kappa(Y'/\Delta(f')) = -\infty$ exactly when

$$\forall i, (r_i - n\alpha_i < 0 \text{ or } r_i - n\alpha_i - n + 2 < 0).$$

Considering this for each i separately, we get the same result as above, ie that $\kappa(Y'/\Delta(f')) = -\infty$ exactly when $\forall i, r_i < \alpha_i$. A similar calculation works for $\kappa = 0, 1, 2$.

6 A good criterion for admissibility

In this section we will prove that a *neat* smooth holomorphic fibration is always admissible.

Definition 6 *A smooth holomorphic fibration $f : X \to Y$ is* neat *if there exists a bimeromorphic holomorphic map $u : X \to X'$ with X' smooth such that every f-exceptional divisor of X is also u-exceptional.*

So, for example, any smooth holomorphic fibration with a meromorphic inverse is neat, wheras the first fibration given in Section 5 is not.

We begin by defining the *sheaf of differential forms defined by a fibration.*

Definition 7 *Let $f : X \to Y$ be a smooth holomorphic fibration. We define the* sheaf of differential forms defined by f *to be*

$$F_f := f^*(K_Y) = f^{-1}(K_y) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

(see [4]).

This is a coherent subsheaf of $\Omega_X^{\dim Y}$.

It follows from standard results on pullbacks of sheaves (see [4]) that F_f is a bimeromorphic invariant - it depends only on the equivalence class of f, and is unaffected by bimeromorphic modifications to either X or Y.

We define $\kappa(F_f)$ to be its Kodaira dimension; this is defined (as for any coherent sheaf) in the following way:

If $h^0\left(X, F_f^{\otimes m}\right) = 0$ for every strictly positive integer m, then set $\kappa(F_f) = -\infty$.

Otherwise, for each positive integer m where $h^0\left(X, F_f^{\otimes m}\right) \neq 0$, let κ_m denote the dimension of the image of the natural map

$$X \to \mathbb{P}\left(H^0\left(X, F_f^{\otimes m}\right)\right),$$

and let $\kappa_m = -\infty$ otherwise. Then let

$$\kappa(F_f) := \max_{m \geq 1} \{\kappa_m\}.$$

which exists because it is bounded above by the dimension of X.

We will explore the relationship between F_f and $f^*(\Delta(f))$, and show that they are similar. The sheaves F_f are in some ways easier to work with, and this will allow us to prove our result, that neatness implies admissibility.

Definition 8 *Given a $\mathbb{Q}$-divisor $D = \sum_i m_i D_i$ on a smooth space X, we define its 'round up' $\lceil D \rceil$ to be*

$$\lceil D \rceil := \sum_i \lceil m_i \rceil D_i.$$

Note that this is NOT well defined up to linear equivalence.

Definition 9 *Let $f : X \to Y$ a smooth holomorphic fibration. We define the sheaf $F(f)$ by*

$$F(f) := f^*(K_Y) \otimes \mathcal{O}_X \left(\lceil f^*(\Delta(f)) \rceil\right)$$

Recalling the definition $\kappa(Y/\Delta(f)) = \kappa(Y, K_Y + \Delta(f))$, this seems a sensible sheaf to define.

For example, let

$$f_0 : \mathbb{P}^1 \to \mathbb{P}^1$$

where $f_0(x : y) = (x^2 : y^2)$,, then let $j : \mathbb{P}^1 \to \mathbb{P}^1$ send $(x : y) \mapsto (-x : y)$. Let E an elliptic curve, and $t : e \to E$ a translation of order 2. Let X be a desingularisation of

$$\frac{\mathbb{P}^1 \times E}{\langle j \times t \rangle}$$

and f be the lift of f_0.

We see $\Delta(f) = \frac{1}{2}(0 : 1) + \frac{1}{2}(1 : 0)$, so

$$f^*(\Delta(f)) = f^{-1}(0 : 1) + f^{-1}(1 : 0)$$

(considering $f^{-1}(x : y)$ as reduced schemes), so $F(f)$ is the sheaf corresponding to the divisor

$$-4.E + 2.E = -2.E$$

.

We note that the irreducible components of $\Delta(f)$ will always have multiplicity less than 1, so the components of $\lceil\Delta(f)\rceil$ will always have multiplicity 1.

We also note that in general

$$f^*(\lceil\Delta(f)\rceil) \neq \lceil f^*(\Delta(f))\rceil :$$

in the example above,

$$\lceil f^*(\Delta(f))\rceil = f^{-1}(0:1) + f^{-1}(1:0)$$

wheras

$$f^*(\lceil\Delta(f)\rceil) = 2.f^{-1}(0:1) + 2.f^{-1}(1:0).$$

We wish to say F_f and $f^*(\Delta(f)$ are similar, so we need a way to say when we can 'ignore' divisors.

Definition 10 *Let $f : X \to Y$ be a smooth holomorphic fibration, and D an effective divisor on X. We say D is* partially supported on the fibres of f *if $f(D) \neq Y$ and if for every irreducible component C of $f(D)$ of codimension 1 in Y, $f^{-1}(C)$ contains an irreducible component which surjects onto C by f but which is not contained in the support of D.*

Our f above maps to $\mathbb{P}^1$, and so this definition can tell us nothing illuminating.

Consider applying the same definition to a map without connected fibres, such as f_0 above. Then it is easy to find a partially supported D by taking only one connected component of a fibre; the irreducible component which surjects onto C by f but which is not contained in the support of D can be taken to be another connected component of the same fibre. Thus in some way this definition measures how close f is to having disconnected fibres, but this is a great oversimplification.

Note also that for a blowup of projective space considered as a fibration, there are no divisors partially supported on the fibres since the exceptional locus never surjects onto a codimension 1 divisor.

If D is partially supported on the fibres of f, so are its positive multiples.

Next we have a proposition which helps to explain the motivation for the definition above, and the sense in which partially supported divisors can be ignored:

Proposition 11 *Let $f : X \to Y$ be a smooth holomorphic fibration, and D a divisor on X partially supported on the fibres of f. Let L be any line bundle (i.e.a locally free $\mathcal{O}(Y)$ bundle of rank 1) on Y. Then the natural map of sheaves $\ell : L \to f_*(F^*(L) + D)$ is an isomorphism.*

Proof:
First we observe that the natural map ℓ is an injection, because f is a surjection.

It suffices to show ℓ is a surjection *locally*, so we can assume L is trivial ($L \cong \mathcal{O}_Y$).

Now because X and Y are compact, and f is surjective, we get that $f_*(F^*(L)) \cong L$. So the proposition is saying that partially supported divisors 'do not add any more' to the sheaf.

We can assume $\mathcal{O}_X(D) \subset f^*(\mathcal{O}_Y(C))$ for some effective divisor C on Y. Local sections of $f^*(\mathcal{O}_Y(C))$ are of the form $f^*(y/c)$ where y is holomorphic on Y and c is a local equation of C. We need to show that c divides y (because then $f^*(\mathcal{O}_Y(C)) \subset \mathcal{O}_Y$, so we would be done).

Local sections of $\mathcal{O}_X(D)$ are meromorphic functions on X of the same form but with poles contained in D. Now $f^*(y/c)$ will in general have poles everywhere on $f^{-1}(C)$, but by assumption we must have that all its poles are contained in D. But D is partially supported on the fibres of f, c divides y and we are done. $\square$

Note that we did not use that f was holomorphic, and in fact the proof works fine without this assumption.

Next we have:

Proposition 12 *Let $f : X \to Y$ a smooth holomorphic fibration. Then there exists a Zariski-closed subset $A \subset Y$ of codimension at least 2 such that $F(f)+D$ and F_f are naturally isomorphic over $X - B := f^{-1}(Y - A)$ for some effective divisor D on X partially supported on the fibres of f.*

Proof:
Recall

$$F(f) = f^*(K_Y) \otimes \mathcal{O}_X\left(\lceil f^*(\Delta(f))\rceil\right).$$

Let

$$A := Sing(Sup(\Delta(f))) \cup \{f(E) : E \text{ an } f\text{-exceptional divisor on } X\}$$

where $Sing(Sup(\Delta(f)))$ is the singular set of the support of $\Delta(f)$.
Outside $f^{-1}(Sup(\Delta(f)))$, we get that

$$F(f)(U) = f^(K_Y)(U) \otimes \mathcal{O}_X(U)$$

where U is an open set such that $U \cap f^{-1}(Sup(\Delta(f))) = \emptyset$.
X and Y are smooth, so we get

$$\begin{aligned} F(f)(U) &= f^*(K_Y)(U) \otimes \mathcal{O}_X(U) \\ &= f^{-1}(K_Y)(U) \otimes \mathcal{O}_X(U) \otimes \mathcal{O}_X(U) \\ &= f^*(K_Y)(U) \\ &= F_f(U). \end{aligned}$$

So we want to show $F(f) + D \cong F(f)$, i.e. that $S \subset F(f)$ (outside $f^{-1}(Sup(\Delta(f)))$). However, a smooth codimension 1 subvariety of a smooth projective variety is locally irreducible, so this is clear.

Now we want to show $F(f) + D \cong F_f$ on $X - B = f^{-1}(Y - A)$. In other words, we want to show $F_f - F(f)$ is partially supported on the fibres of f over $X - B$. We have just seen that this holds outside of $\Delta(f)$. It remains to check it on the smooth locus of $\Delta(f)$.

For every component Δ_i of the support of $\Delta(f)$, we write

$$D_i = \sum_{j \in J_i} m_{i,j} D_{i,j}$$

for the union of all components of $f^*(\Delta_i)$ mapped surjectively onto Δ_i by f. Then the multiplicity of Δ_i is

$$m_i = inf\left\{m_{i,j} : j \in J_i\right\}.$$

We want to consider what happens near a smooth point of some Δ_i not in A.

At a general point of $D_{i,j}$ choosing the correct coordinates
$(x_1, \dots x_n)$ local coordinates on X, and
$(y_1, \dots y_p)$ local coordinates on Y, we can write

$$f((x_1, \dots x_n)) = (x_1^{m_{i,j}}, x_2, \dots x_p).$$

Now

$$F(f) = f^*(K_Y) \otimes \mathcal{O}_X\left(\lceil f^*(\Delta(f)) \rceil\right)$$

and

$$F_f = f^*(K_Y),$$

so it remains to show $\lceil f^*(\Delta(f)) \rceil$ is partially supported on the fibres of f over $X - B$.

Now

$$f^*(\Delta(f)) = x_1^{\frac{m_{i,j}}{m_i}} dx_i \wedge \dots \wedge dx_p.$$

So it remains to see that the sheaf generated in $\mathcal{O}_X$ by $x_1^{\left\lfloor -\frac{m_{i,j}}{m_i} \right\rfloor}$ is partially supported on the fibres of f over $X - B$.

$$\begin{aligned} f\left(x_1^{\left\lfloor -\frac{m_{i,j}}{m_i} \right\rfloor}\right) &= \left(y_1^{\frac{1}{m_{i,j}}}\right)^{\left\lfloor -\frac{m_{i,j}}{m_i} \right\rfloor} \\ &= y_1^{\left\lfloor -\frac{1}{m_i} \right\rfloor} \\ &= y_1^{-1} \end{aligned}$$

So let $C : (y_1 = 0)$ (the only thing it can be, locally).

So $f^{-1}(C) : (x_1^{m_{i,j}} = 0)$.

Noting $m_{i,j} \geq 2$, and that there exist $m_{i,j}$ distinct roots of $\left(x_1^{m_{i,j}}\right)$, we can just take a root not in D, and we are done. □

Now we combine the last two results to give:

Proposition 13 *Let $f : X \to Y$ be a smooth holomorphic fibration. Let m be a strictly positive integer with sufficiently many factors. Then:*

1) The natural isomorphism between $F(f) + D$ and F_f over $X - B$ gives a natural injection

$$\varphi : H^0\left(X, F_f^{\otimes m}\right) \hookrightarrow H^0\left(X, F(f) + D\right) \cong H^0\left(Y, m.(K_Y + \Delta(f))\right)$$

2) If f is neat, φ is injective.

Proof:

1) By Proposition 11, we get that

$$H^0\left(X, F(f) + D\right) \cong H^0\left(Y, m \cdot (K_Y + \Delta(f))\right)$$

holds.

Then by Proposition 12,

$$H^0\left(X - B, F_f^{\otimes m}\right) \cong H^0\left(X - B, F(f) + D\right).$$

B has codimension at least 2 in X and $m.(K_Y + \Delta(f))$ is locally free on Y, so by Hartog's theorem, this extends to an injection

$$\varphi : H^0\left(X, F_f^{\otimes m}\right) \hookrightarrow H^0\left(X, F(f) + D\right)$$

.

2)Neatness means that a holomorphic map $u : X \to X'$ to smooth X' sending the f-exceptional divisors to codimension at least 2 divisors on X', then $u(B)$ has codimension at least 2 in X' (either B was f-exceptional, so shrunk by u, or not in which case it already had codimension at least 2 in X).

Then any section in $H^0\left(X - B, F(f) + D\right)$ extends to a section of $F_f^{\otimes m}$ over X, so we have an isomorphism as claimed. □

This has the following useful corollary, which is the main result of this section.

Proposition 14 *Let $f : X \to Y$ smooth holomorphic fibration. Then:*

1) $\kappa(F_f) \leq \kappa(Y/\Delta(f))$

2) $\kappa(F_f) = \kappa(Y/\Delta(f))$ if f is neat.

3) $\kappa(F_f) = \kappa(f)$

Proof:

(1) follows from Proposition 13 (1) and the alternative definition of Kodaira dimension as

$$\kappa(D) = \limsup_{m \to \infty} \frac{\log h^0\left(X, \lfloor mD \rfloor\right)}{\log m}$$

for D a $\mathbb{Q}$-divisor (see [1] , page 16).

(2) similarly follows from Proposition 13 (2).

To get (3), take a neat admissible model f' of f, then $\kappa(F_f) = \kappa(Y'/\Delta(f')) = \kappa(f)$. □

Thus any smooth neat holomorphic fibration is admissibly. In particular, fibrations to curves are admissible, but as you can see, this is a hard way to get that result.

7 Appendix: calculations for Example 3.7

We claimed in Section 3.7 that the surface $X \subset \mathbb{P}^2_{(X:Y:Z)} \times \mathbb{P}^1_{(s:t)}$ defined by the vanishing of:

$$F(X,Y,Z,s,t) := \frac{st}{2}(s^{m-2}+t^{m-2})((X+Y+Z)^d - Y^d - Z^d) + s^m Y^d + t^m Z^d$$

where $m \geq 3, d \geq 4$ has singularities exactly as follows:

• the fibre[8] over $(0:1)$ is singular at the point defined by $Z = 0, (X+Y)^d = Y^d$. The exceptional locus is a bunch of $\mathbb{P}^1$s.

• the fibre over $(1:0)$ is similar to the one above by symmetry, with isomorphic exceptional locus.

• the fibre over $(1:1)$ is singular only at the points defined by $X+Y+Z+0, Y^d = Z^d$, with exceptional locus a bunch of $\mathbb{P}^1$s again.

• X is smooth outside these fibres.

In this appendix we prove these assertions by calculation.

A note on the notation used for blowups: if we start with an affine equation in a, b, c, the blowup at the origin will be the variety inside

$$\mathbb{A}^3_{(a,b,c)} \times \mathbb{P}^2_{(a_1:b_1:c_1)}$$

cut out by

$$rank \begin{pmatrix} a & b & c \\ a_1 & b_1 & c_1 \end{pmatrix} \leq 1.$$

First we calculate the partial derivatives of F with respect to X, Y, Z, s, t:

$$\frac{\partial F}{\partial X} = \frac{st}{2}(s^{m-2}+t^{m-2})d(X+Y+Z)^{d-1} \tag{71}$$

$$\frac{\partial F}{\partial Y} = \frac{st}{2}(s^{m-2}+t^{m-2})\left(d(X+Y+Z)^{d-1} - dY^{d-1}\right) + ds^m Y^{d-1} \tag{72}$$

$$\frac{\partial F}{\partial Z} = \frac{st}{2}(s^{m-2}+t^{m-2})\left(d(X+Y+Z)^{d-1} - dZ^{d-1}\right) + dt^m Z^{d-1} \tag{73}$$

$$\frac{\partial F}{\partial s} = \frac{1}{2}((m-1)ts^{m-2}+t^{m-1})\left((X+Y+Z)^d - Y^d - Z^d\right) + ms^{m-1}Y^d \tag{74}$$

$$\frac{\partial F}{\partial t} = \frac{1}{2}((m-1)st^{m-2}+s^{m-1})\left((X+Y+Z)^d - Y^d - Z^d\right) + mt^{m-1}Z^d \tag{75}$$

We prove the last assertion first, namely that X is smooth outside the fibres over $(s:t) = (0:1), (1:0)$ or $(1:1)$:

[8] of the fibration to the $\mathbb{P}^1$ with coordinates $(s:t)$

Suppose $(X : Y : Z)(s : t)$ is a singular point on X outside the fibres listed.

(71) shows us that $X + Y + Z = 0$ or $s^{m-2} + t^{m-2} = 0$. If $s^{m-2} + t^{m-2} = 0$, then (72) and (73) together show that $Y = Z = 0$, so $X \neq 0$, and then considering (74) and (75) shows that $st^{m-2} = 0$.

If $X + Y + Z = 0$, then we can get from (72) and (73) that either $Y = 0$ or $Z = 0$, and clearly not both. Without loss of generality assume $Y = 0$, so $(m-1)ts^{m-2} + t^{m-1} = 0$. Then (73) and (75) say that either $s = 0$ or $t = 0$, so we are done.

Next we show the first assertion, that the fibres over $(0 : 1)$ is singular at at most the points defined by $X + Z = 0$, with exceptional locus $\mathbb{P}^1$ (noting that the second assertion will then follow by symmetry):

First we will show quite easily where the singularities are. Much harder will be showing that the exceptional loci of these singularities consist only of $\mathbb{P}^1$s.

Setting $s = 0$ in the equations for the partial derivatives above, we get:

$$\frac{\partial F}{\partial X} = 0 \tag{76}$$

$$\frac{\partial F}{\partial Y} = 0 \tag{77}$$

$$\frac{\partial F}{\partial Z} = dt^m Z^{d-1} \tag{78}$$

$$\frac{\partial F}{\partial s} = \frac{1}{2}t^{m-1}\left((X+Y+Z)^d - Y^d - Z^d\right) \tag{79}$$

$$\frac{\partial F}{\partial t} = mt^{m-1}Z^d. \tag{80}$$

Now $s = 0$ so $t \neq 0$ so $Z = 0$ and $(X + Y)^d - y^d = 0$ as required.

Next to analyse the exceptional locus to this singularity: all we need to know is that the exceptional locus consists of a bunch of $\mathbb{P}^1$s. For this it is sufficient (though not necessary; consider the case of three copies of $\mathbb{P}^1$) to show that the equation defining X in the neighbourhood of the singularity has some non-zero part in total degree less than or equal to 2 (since when you blow up, you can ignore all of the equation except the part in lowest degree).

For simplicity we will look at the singularity at $X = 0$ - this is equivalent to any other by an analytic change of coordinates. Setting $Y = t = 1$, we get the equation

$$F'(X, Z, s) = \frac{s}{2}(s^{m-2} + 1)\left((X + 1)^d - Z^d - 1\right) + s^m + Z^d.$$

For $m, d \geq 4$, the degree 2 part is simply sX, so the exceptional locus is a pair of $\mathbb{P}^1$s. Transforming F' to the piece of the affine blowup on $z_1 \neq 0$, we get the equation

$$\bar{F'}(X_1, Zs_1) = \frac{s_1}{2}(s_1^{m-2}Z^{m-2}+1)(X_1^d Z^{d-1}+\ldots ZX_1^2+X_1-Z^d)+s_1^m Z^{m-2}+Z^{d-2}$$

which is singular only at $(0, 0, 0)$, the origin in that affine part of the blowup as long as m, d are sufficiency large. (s_1 and X_1 can be thought of as coordinates on the inserted $\mathbb{P}^2$, and Z as the coordinate of the starting affine space 'sticking out of the $\mathbb{P}^2$'). (There remain 2 more points on the exceptional locus to check for nonsinglarity, but this is straightforward). Thus if m, d are sufficiently large, it is easy to see we can repeat this process of blowing up at the origin to obtain an exceptional locus which is a bunch of $\mathbb{P}^1$s.

It only remains to see what happens when m, d are not so large: this includes both the cases where $m \leq 3$ to start with, and where successive blowups have reduced them. Note that successive blowups do not exactly just reduce m, d, but the effect is similar, and that the result is the same is easy to check; we will thus only do the cases $m = 2$ and $m = 3$ in the original equation

$$F'(X, Z, s) = \frac{s}{2}(s^{m-2} + 1)\left((X + 1)^d - Z^d - 1\right) + s^m + Z^d.$$

Here it is easy to see that the degree ≤ 2 part is $sX + s^2$ if $m = 2$ and sX if $m = 3$, and the induction proceeds much as before, but now

$$\bar{F'} = s_1(X_1^d Z^{d-1} + \ldots + ZX_1^2 + X_1 - Z^{d-1}) + s_1^2 + Z^{d-2}$$

and the induction only reduces d. When $d = 3$, we get

$$\bar{F} = s_1(X_1^3 Z^2 + X_1^2 Z + X_1 - Z^2) + s_1^2 + Z$$

so the result is non-singular, and when $d = 2$, we get

$$\bar{F} = s_1(X_1^2 Z + X_1 - Z) + s_1^2 + 1$$

which is again non-singular.

Thus we see that on the fibre over $s = 0$, the only singularities have exceptional locus a bunch of $\mathbb{P}^1$s.

It remains to consider the fibre over $s = t$. This case will in fact be simpler because only two blowups will be required to resolve the singularity, whatever the values of m, d.

Taking $s = t = 1$, we see from (71) that $X + Y + Z = 0$, and then from (74) that $Z^d - Y^d = 0$. Again for simplicity we change coordinates to consider the case $Y = 1, Z = 1, X = -2$. Changing coordinates to put this at the origin of $\mathbb{A}^3_{(Y,Z,t)}$ (setting $s = X = 1$) we get the equation

$$F(Y, Z, t) = \frac{1}{2}\left(1 + (t-1)^{m-2}\right)(t-1)\left((Y+Z)^d - (Y-1)^d - (Z-1)^d\right) + (Y-1)^d + (t-1)^m (Z-1)^d \tag{81}$$

To find the exceptional locus of the blowup, we look for the linear part of (81): if m is odd this is given by

$$dY - dZ - (m-2)t$$

and if m is even, it is given by

$$t$$

both of which represent a $\mathbb{P}^1$.

Next we look for singularities on the exceptional locus of the blowup; we will work on the piece $Y_1 \neq 0$, but it is quite easy to check the others behave similarly. The equation we get on the affine piece $Y_1 \neq 0$ is

$$F'(Y, Z_1, t_1) =$$

$$\frac{1}{2}\left(1+(Yt_1-1)^{m-2}\right)(Yt_1-1)\left(Y^d(Z_1+1)^d-(Y-1)^d-(YZ_1-1)^d\right)+(Y-1)^d+(Yt_1-1)^m(YZ_1-1)^d$$

Now the derivative of this with respect to Z_1 and t_1 vanishes everywhere on the exceptional $\mathbb{P}^2$, but the derivative with respect to Y is more interesting:

$$\left.\frac{\partial F'}{\partial Y}\right|_{Y=0} = \begin{cases} (-1)^{d-1}\left(\frac{t_1 m}{2}+1+d-Z_1 d\right) & \text{if } m \text{ odd} \\ \frac{(-1)^{d-1}}{2}(4d+3t_1 m) & \text{if } m \text{ even} \end{cases}$$

Now if m is even, the exceptional locus is given by $t_1 = 0$, and so **the result is non-singular**.

Thus it remains to see what happens when m is odd; in this case the exceptional locus is given by

$$d(Y-Z)-(2m-2)t = 0$$

and this meets the singular locus at

$$\begin{aligned} Y &= 0 \\ t_1 &= \tfrac{2}{4-5m} \\ Z_1 &= \tfrac{5dm-4d+4m-4}{5dm-4d} \end{aligned}$$

If we let a denote $\frac{2}{4-5m}$ and b denote $\frac{5dm-4d+4m-4}{5dm-4d}$, we want to translate this to the origin, then repeat the steps above. The translation gives the equation

$$\begin{aligned} &F''(Y, Z_1 t_1) = \\ &\tfrac{1}{2}\left(1+(Yt_1-aY-1)^{m-2}\right)(Yt_1-aY-1) \\ &\left(Y^d(Z_1-b+1)^d-(Y-1)^d-(YZ_1-bY-1)^d\right)+(Y-1)^d+(Yt_1-aY-1)^m(YZ_1-bY-1)^d. \end{aligned}$$

This has linear part

$$Y(am-2a \pm d \mp a \pm b)$$

(where the coefficient is non zero), and hence the exceptional locus is given by $Y = 0$. The transformed equation for X in the vicinity of the affine piece $t_1 \neq 0$ is then given by

$$\begin{aligned} &F''(Y_1, Z_1, t) = \\ &\tfrac{1}{2}\left(1+(Y_1t^2-aY_1t-1)^{m-2}\right)(Y_1t^2-aY_1t-1) \\ &\left(t^dY_1^d(tZ_1-b+1)^d-(tY_1-1)^d-(t^2Y_1Z_1-btY_1-1)^d\right) \\ &+(tY_1-1)^d+(Y_1t^2-atY_1-1)^m(t^2Y_1Z_1-btY_1-1)^d, \end{aligned}$$

the derivatives of which with respect to Y_1, s and t do not vanish simultaneously on the line $Y_1 = 0$, so we are done.

References

[1] Caucher Birkar. Birational geometry. `http://www.dpmms.cam.ac.uk/%7Ecb496/birgeom.pdf`, 2007.

[2] Anita Buckley. Orbifold riemann-roch for threefolds with an application to calabi-yau geometry. `arXiv:math/0309033v1`, 2003.

[3] Frederic Campana. Special varieties and classification theory. *ANNALES DE L'INSTITUT FOURIER*, 54:499, 2004.

[4] Robin Hartshorne. *Algebraic Geometry*. Springer, Berlin, 1977.

[5] Janos Kollar. *Rational Curves on Algebraic Varieties*. Springer, Berlin, 1996.

[6] Janos Kollar. *Lectures on Resolution of Singularities*. Princeton University Press, Princeton and Oxford, 2007.

[7] S. Y. Lu. A refined kodaira dimension and its canonical fibration. `http://arxiv.org/abs/math/0211029`, 2002.

[8] Miles Reid. *Undergraduate Algebraic Geometry*. Cambridge University Press, Cambridge, UK, 1988.

[9] Miles Reid. *Chapters On Algebraic Surfaces*. TBC, TBC, 2008.

[10] Sandor Kovacs Stefan Kebekus. Families of varieties of general type over compact bases. `arXiv:0704.2556v1`, 2007.

[11] Ravi Vakil. Complex algebraic surfaces. `http://math.stanford.edu/~vakil/02-245/index.html`, 2002.

[12] E. Viehweg. Positivity of direct image sheaves and applications to families of higher dimensional manifolds. ICTP-Lecture Notes 6 (2001) 249-284, 2001.