Invariant Manifolds and Applications for Functional Differential Equations of Mixed Type

Hermen Jan Hupkes

Universiteit Leiden

(Joint work with S.M. Verduyn Lunel and E. Augeraud-Véron)

Mixed Type Functional Differential Equations (MFDEs)

Prototype MFDE is the differential equation given by

$$\dot{x}(t) = f(x(t)) + x(t-1) + x(t+1).$$

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Prototype MFDE is the differential equation given by

$$\underbrace{\underbrace{\dot{x}(t) = f(x(t))}_{\text{ODE}} + x(t-1)}_{\text{DelayEquation}} + x(t+1).$$

- Delay equations have been used for more than half a century.
- They arise naturally in many modelling applications.
- Appropriate functional-analytic setup developed in past two decades, finally enabling extensions of classic ODE results (stable / unstable manifolds, Hopf bifurcations, etc.)
- Recently theory for linear MFDEs has started to develop (Mallet-Paret and Verduyn-Lunel (2001), Härterich et al. (2002))

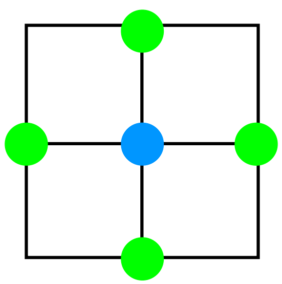
Lattice equations

Main (historical) motivation for study of MFDEs comes from lattice differential equations, e.g., the discrete reaction-diffusion equation

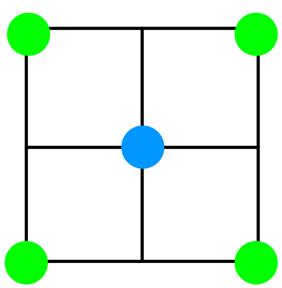
$$\dot{u}_{i,j} \quad = \quad \alpha L_D u_{i,j} - f(u_{i,j}), \quad (i,j) \in \mathbb{Z}^2.$$

L_D is a discrete Laplacian, which could be given by

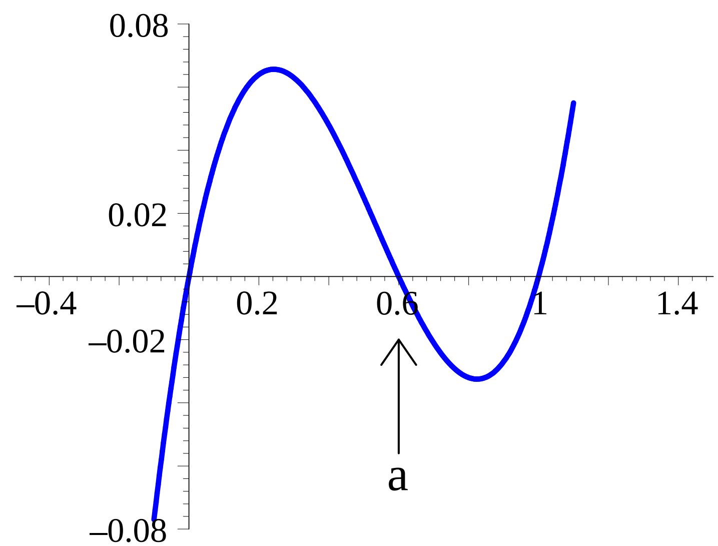
$$L_D u_{i,j} = (\Delta^+ u)_{i,j} \equiv u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}.$$



Discrete Laplacian Δ^+.



Discrete Laplacian $\Delta^\times$.



Bistable nonlinearity, typically

$$f_{\text{cub}}(u) = u(u-a)(u-1).$$

Lattice equations: Anisotropy

We consider travelling wave solutions which propagate at an angle θ.

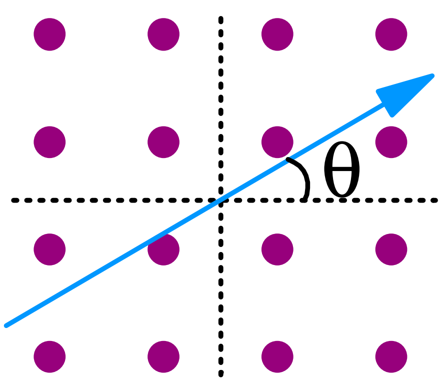
$$u_{i,j}(t) = \phi(i\cos\theta + j\sin\theta - ct).$$

We require $\phi(-\infty) = 0$ and $\phi(\infty) = 1$. Substitution yields the MFDE

$$-c\phi'(\xi) \quad = \quad \alpha L_\theta(\phi)(\xi) - f_{\mathrm{cub}}(\phi(\xi), a)$$

where

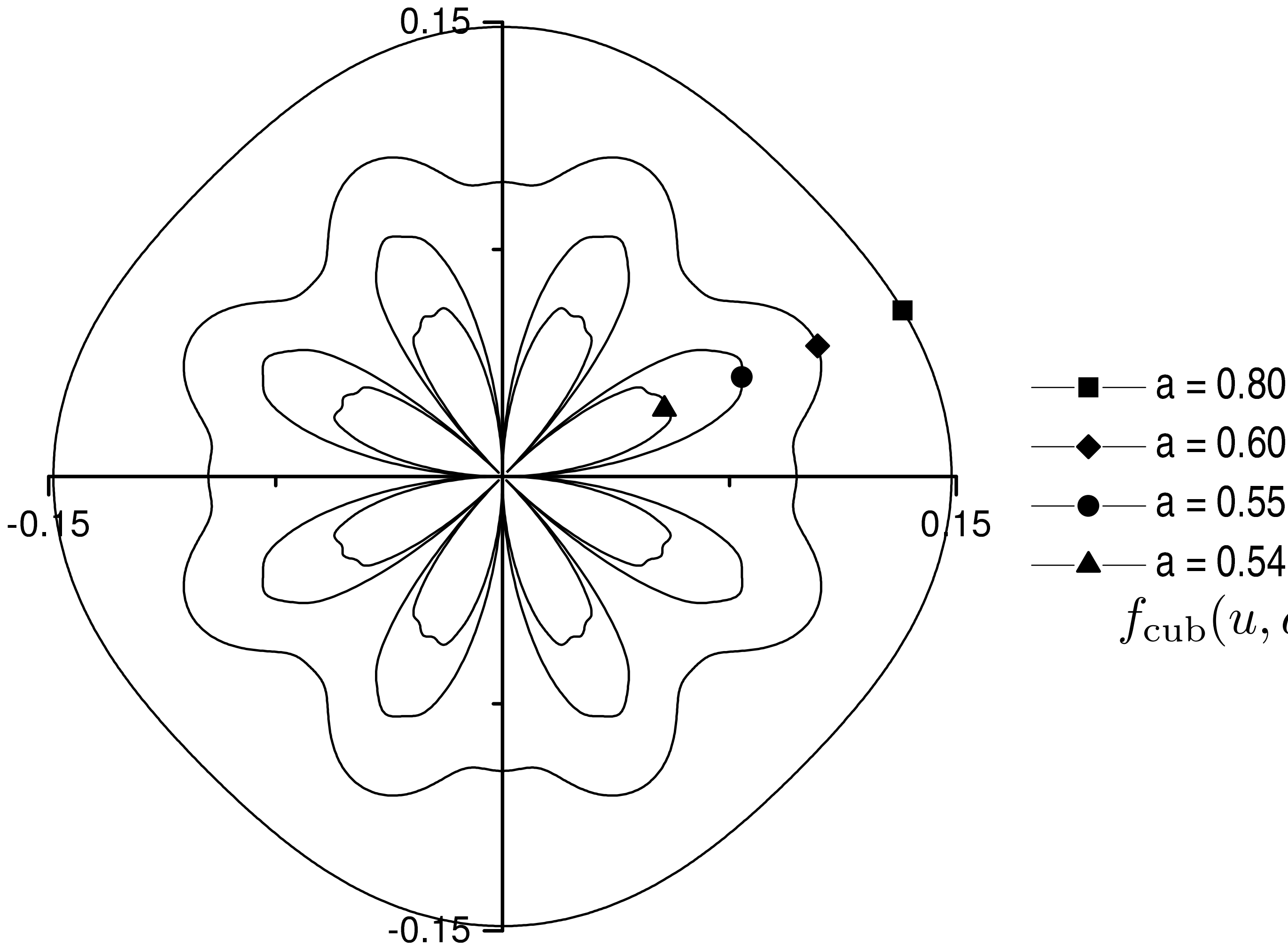
$$L_\theta(\phi) = \phi(\xi + \cos\theta) + \phi(\xi - \cos\theta) + \phi(\xi + \sin\theta) + \phi(\xi - \sin\theta) - 4\phi(\xi).$$



- Notice the θ - dependence in the discrete case, which is absent in PDE case.
- Propogation direction dependent!

Lattice equations: Anisotropy

The lattice anisotropy can be illustrated by studing the $c(\theta)$ relation. Example LDE: $\dot{u}_{i,j} = \frac{1}{4}\big((\Delta^+ u)_{i,j} + (\Delta^\times u)_{i,j}\big) - 10 f_{\mathrm{cub}}(u_{i,j}, a)$.



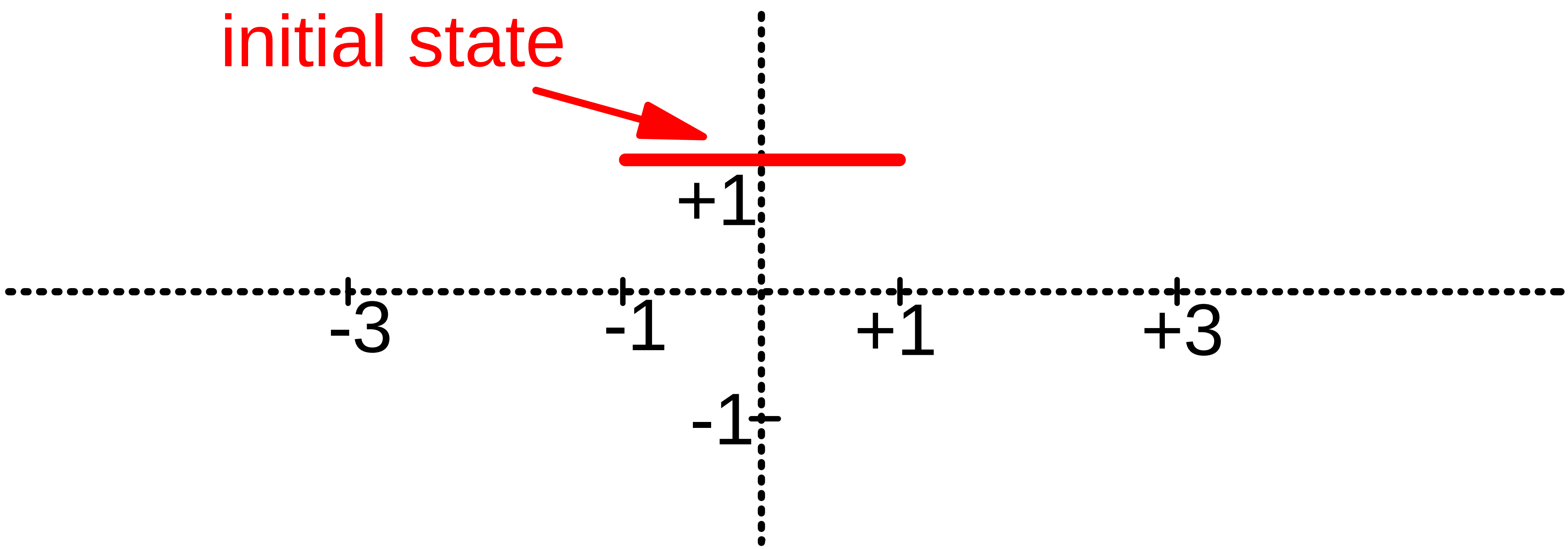
$$f_{\mathrm{cub}}(u,a) = u(u-a)(u-1)$$

- Propagation failure.
- Anisotropy.

MFDEs vs Delay Equations

Consider the MFDE

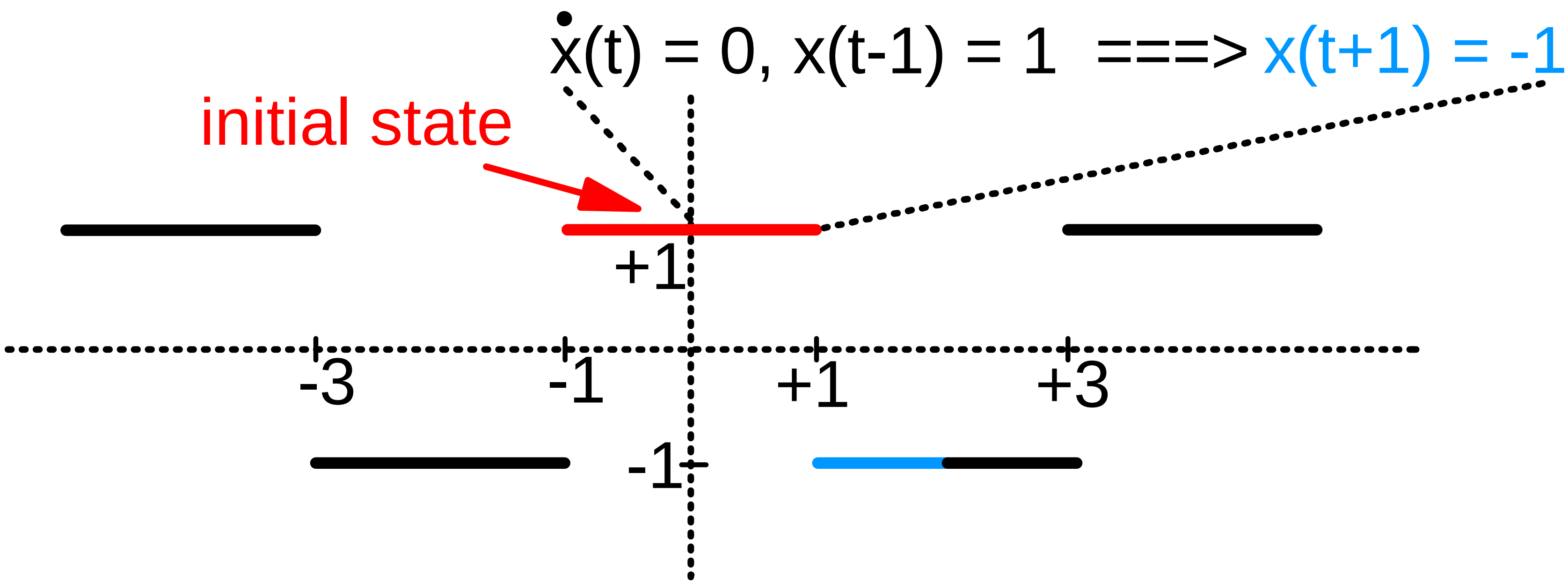
$$\dot{x}(\xi) = x(\xi - 1) + x(\xi + 1).$$



MFDEs vs Delay Equations

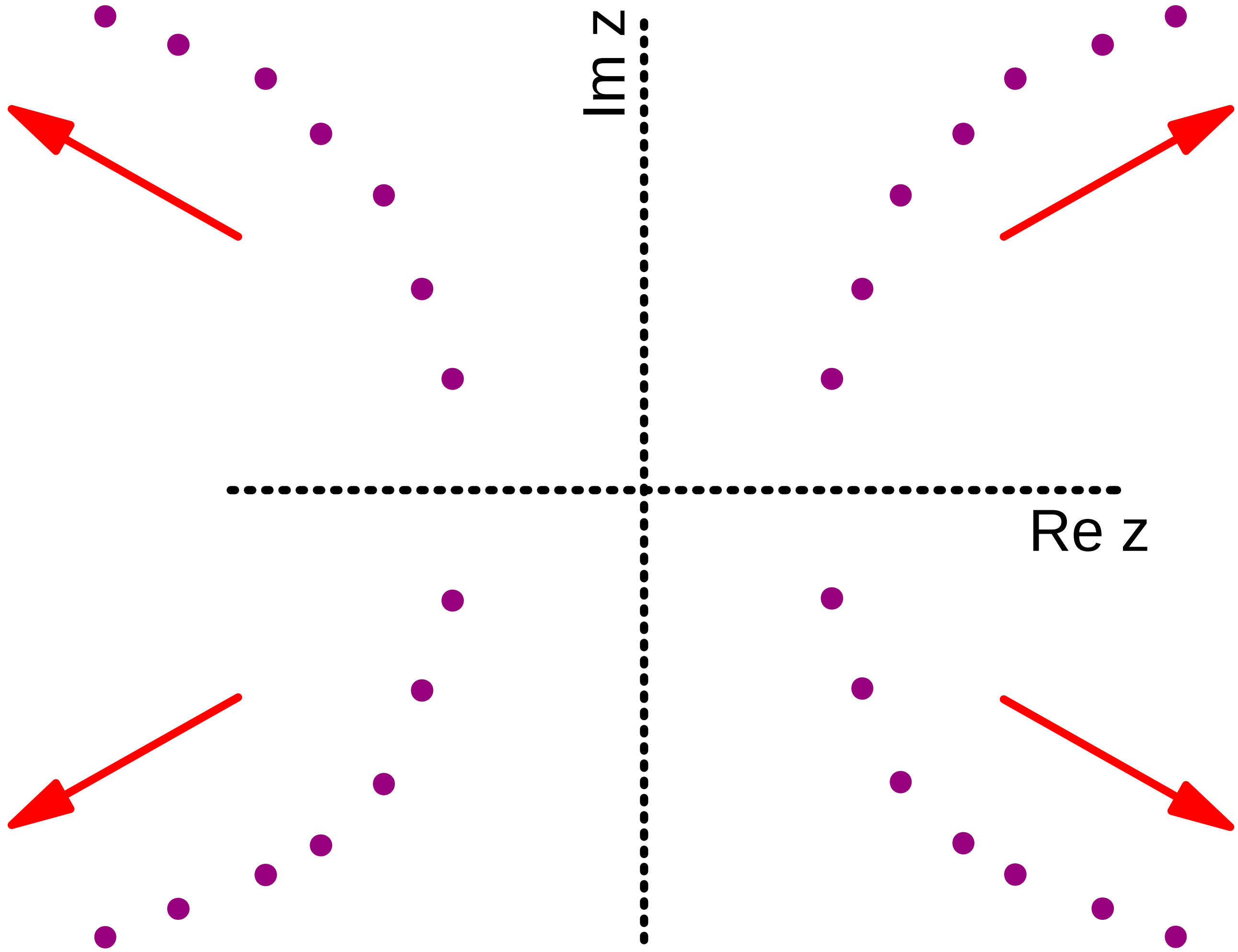
Consider the MFDE

$$\dot{x}(\xi) = x(\xi - 1) + x(\xi + 1).$$



- Continuity lost $\Longrightarrow$ problem not well-defined $\Longrightarrow$ no semi-group techniques.

MFDE: What is going on?



Substitution of $e^{z\xi}$ into

$$\dot{x}(\xi) = x(\xi - 1) + x(\xi + 1),$$

yields the characteristic equation

$$\Delta(z) := z - e^{-z} - e^{z} = 0.$$

- The problem is infinite dimensional (as for delay equations).
- There is no exponential bound possible for solutions, at both $\pm\infty$ (unlike delay equations)!

MFDE: Nonlinear

Now include nonlinear term

$$\dot{x}(\xi) = A_- x(\xi - 1) + A_+ x(\xi + 1) + f(x(\xi)),$$

with $f(0) = Df(0) = 0$, so $x \equiv 0$ is an equilibrium solution.

For applications we are interested in bounded solutions near the equilibrium.

Main Result (H. + Verduyn-Lunel, 2006)

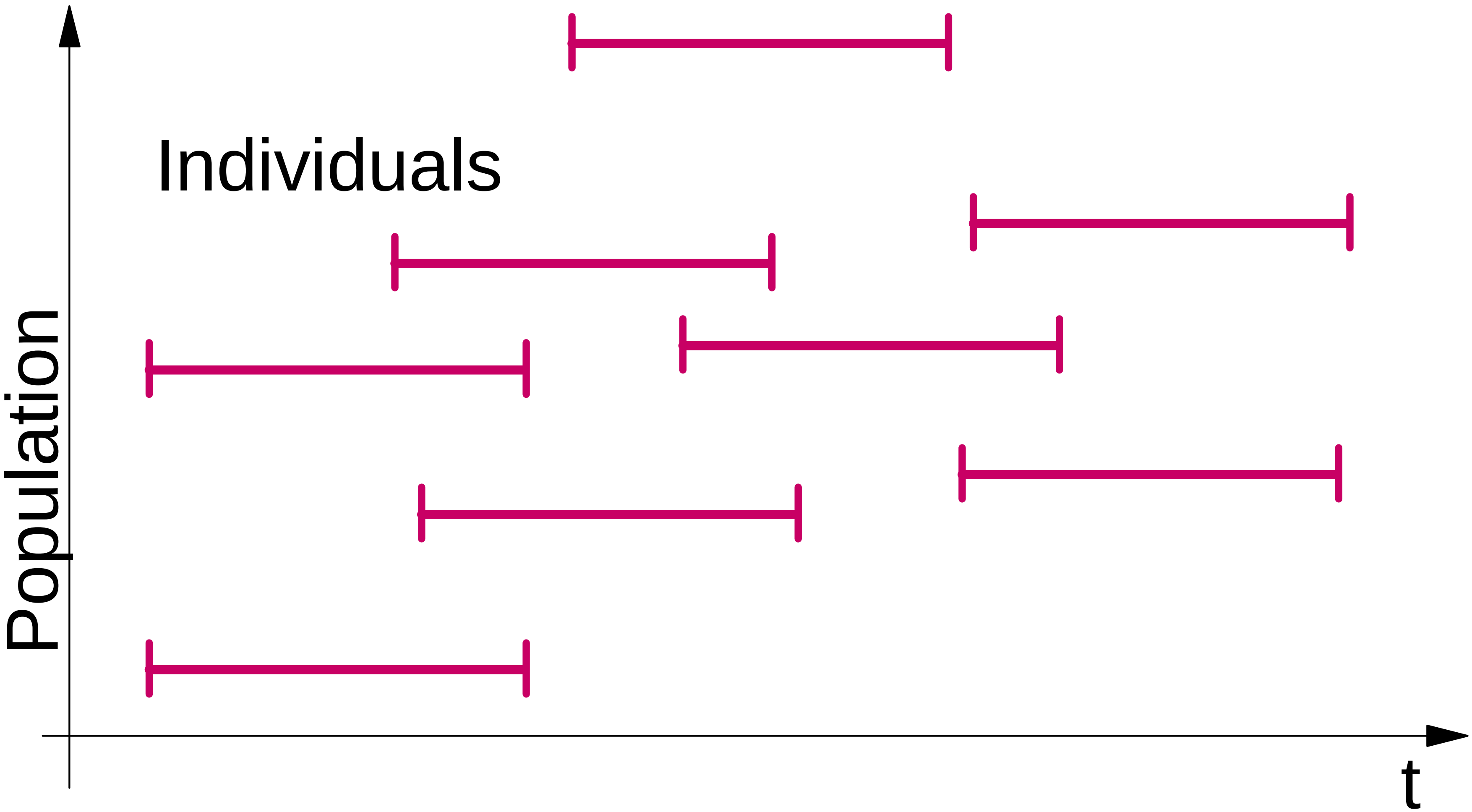
In the neighbourhood of the equilibrium, the dynamics of the MFDE can be completely captured on a finite dimensional object, called the center manifold. The dynamics on this manifold is described by an ODE, which can be explicited calculated.

This reduction

$$\begin{array}{ccc} \text{MFDE} & \longrightarrow & \text{ODE} \\ \infty \, \text{dim} & & n \, \text{dim} \end{array}$$

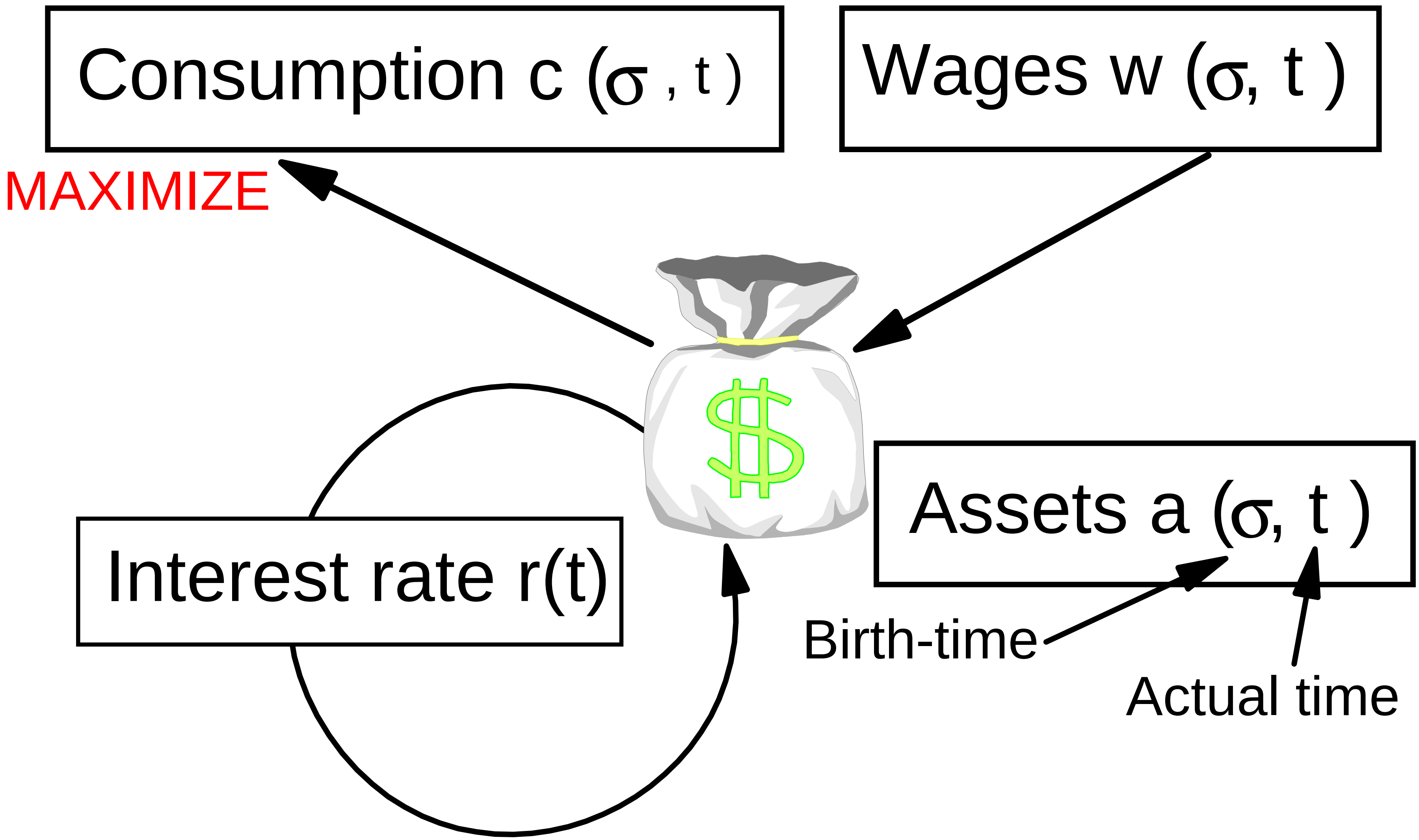
opens up the full toolbox available for ODEs!

Overlapping Generations



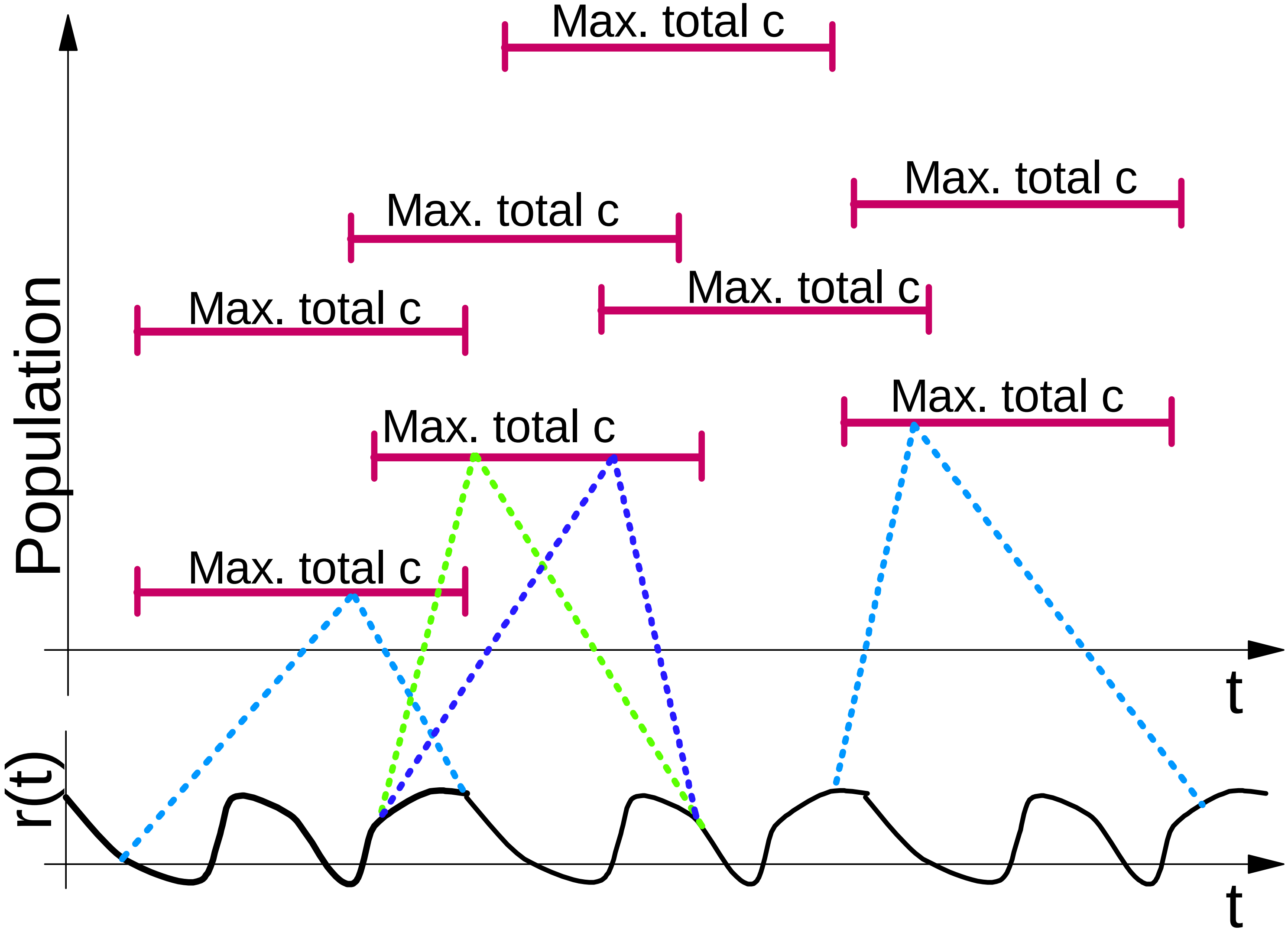
- Proposed by H. d'Albis + E. Augeraud-Véron (2005).
- Population consists of a continuum of individuals, that have fixed lifespan T_{life}.

Overlapping Generations: The Individual



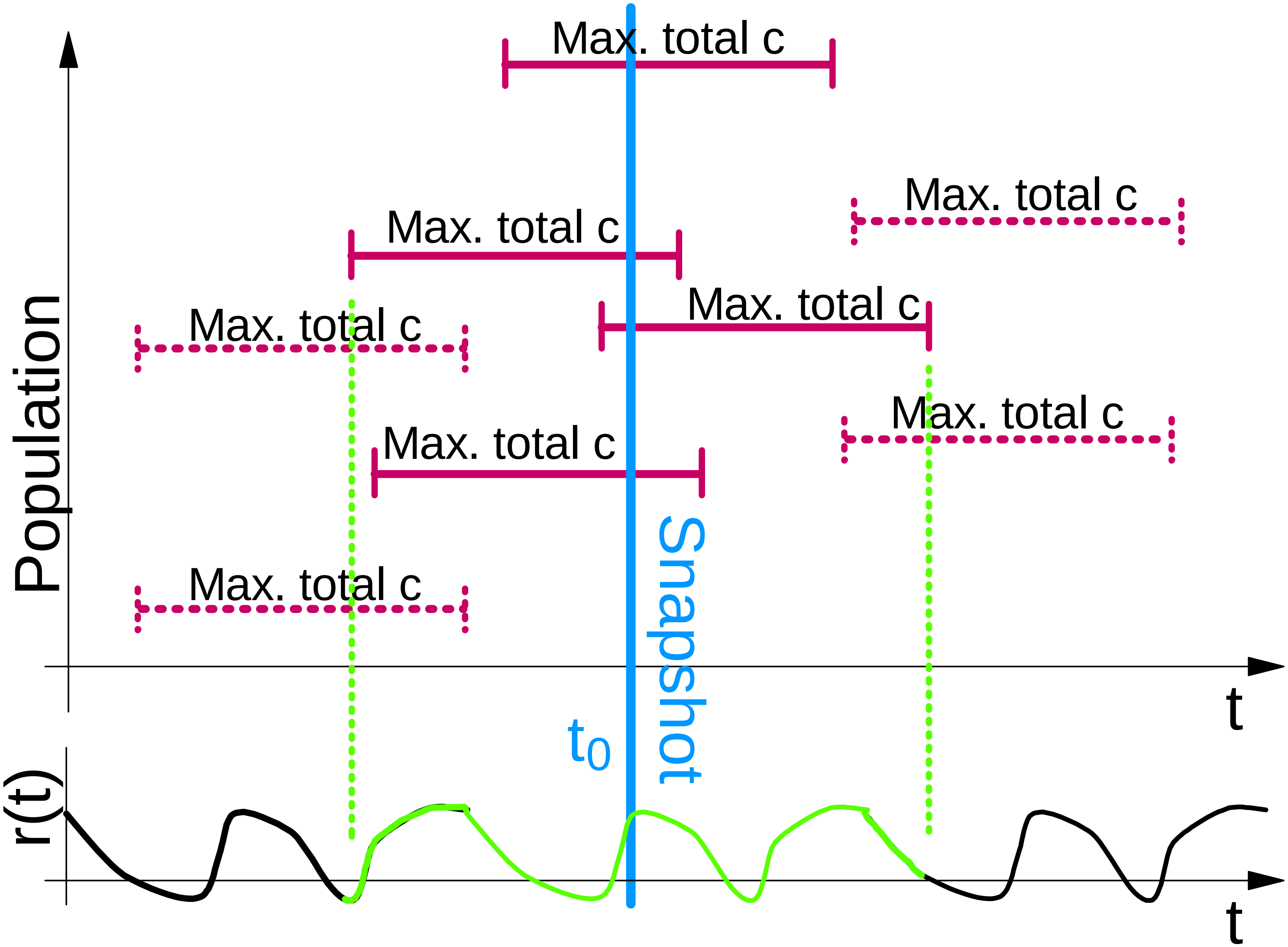
- Each individual wishes to maximize his lifetime consumption, but may not die in debt.

Overlapping Generations: The Collective



- Assumption: Perfect foresight. Consumer can accurately predict interest rate.

Overlapping Generations: The Dynamics



- Equilibrium at t_0 requires summation over all living indv (blue line).
- Condition for $r(t_0)$ involves all values $r(t_0 + \theta)$ for $\theta = -T_{\text{life}} \ldots T_{\text{life}}$.

Overlapping Generations: The Mathematics

Equilibrium condition for the interest rate given by

$$1 = \int_{t-T_{\text{life}}}^{t} \frac{\int_s^{s+1} \exp[-\int_t^v r(u)du]dv}{\int_s^{s+T_{\text{life}}} \exp[-(1-\sigma)\int_t^v r(u)du]dv} ds.$$

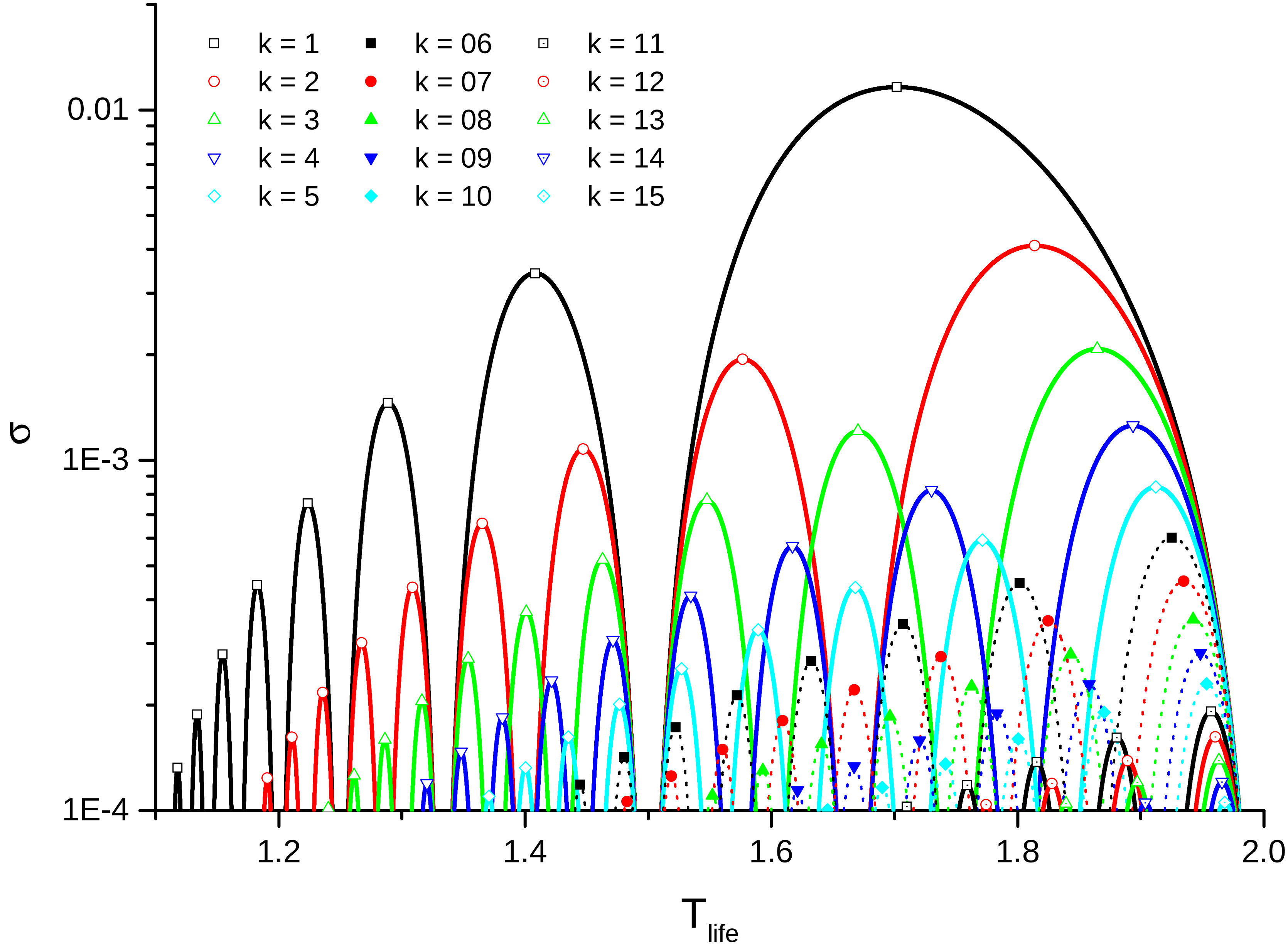
Economists are very interested in periodic solutions!

- Twofold differentiation leads to MFDE.
- Linearization leads to characteristic equation (with $T = T_{\text{life}}$)

$$\Delta(z) \sim [-T\omega e^z + (1-\sigma)e^{zT} + (\omega e^z - T + 1 - \sigma)e^{-zT} + (T - 2 + 2\sigma) + \sigma T^2 z^2].$$

- Look for pairs (σ_0, T_0) such that $\Delta(i\omega) = 0$ for some $\omega \in \mathbb{R}$.
- At these pairs Hopf bifurcations occur, producing periodic orbits for (σ, T) near (σ_0, T_0), with period $T_{\text{period}} \approx \frac{2\pi}{\omega}$.

Overlapping Generations: Hopf bifurcations



Period of periodic orbit $T_{\text{period}} = T_0(T_{\text{life}})/k$.

Overlapping Generations

One might worry about the differentiation needed to turn equilibrium condition into MFDE.

$$1 = \int_{t-T_{\text{life}}}^{t} \frac{\int_s^{s+1} \exp[-\int_t^v r(u)du]dv}{\int_s^{s+T_{\text{life}}} \exp[-(1-\sigma)\int_t^v r(u)du]dv} ds.$$

- Question: how do original equation and associated MFDE relate?
  - Can we lift solutions of MFDE back to original equation?
  - Can we still perform reduction to an ODE?

- H+VL+AV (2007): Can construct center manifold directly for original equation, again (locally) reducing dynamics to an ODE.

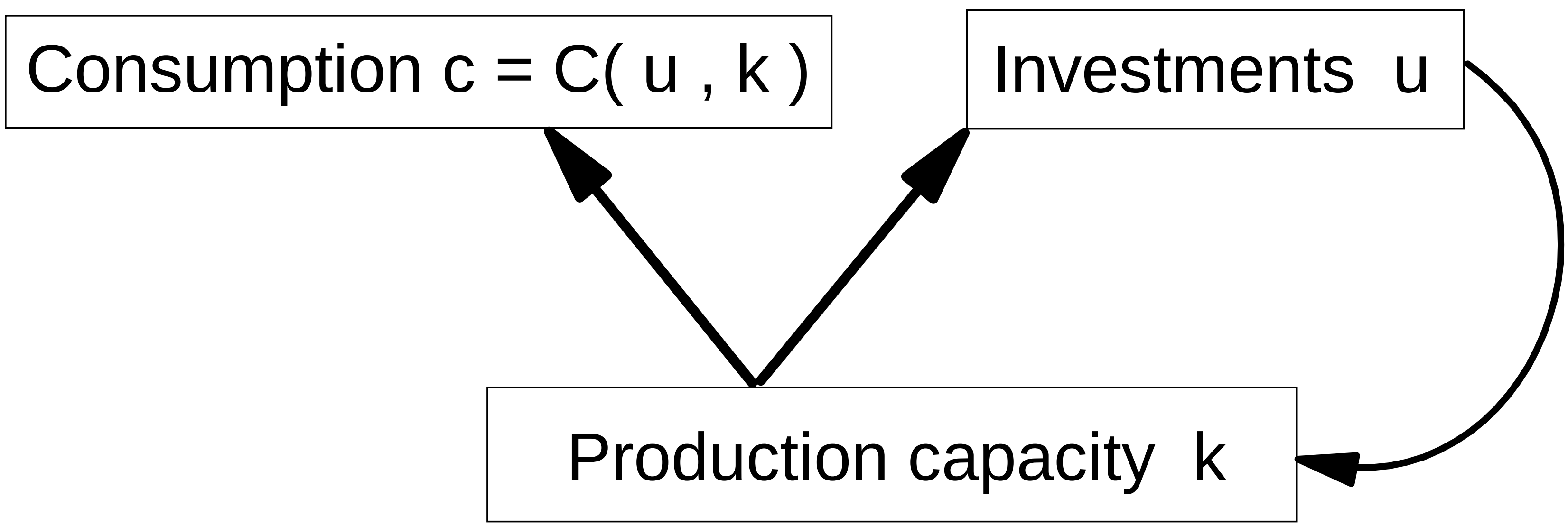
- This should be compared to theory of DAEs (Differential-Algebraic equations), where ODEs are coupled to algebraic equations.

Overview

- MFDEs arise naturally in different disciplines.
  - Indirectly, through travelling wave solutions to lattice equations.
  - Indirectly, through optimal control problems with delays.
  - Directly, through economic overlapping-generation models.
- MFDEs are infinite dimensional problems.
- Near-equilibrium dynamics can be reduced to an ODE on a center manifold.
- Can invoke ODE theory to study periodic orbits etc.
- Works for algebraic mixed type equations as well.

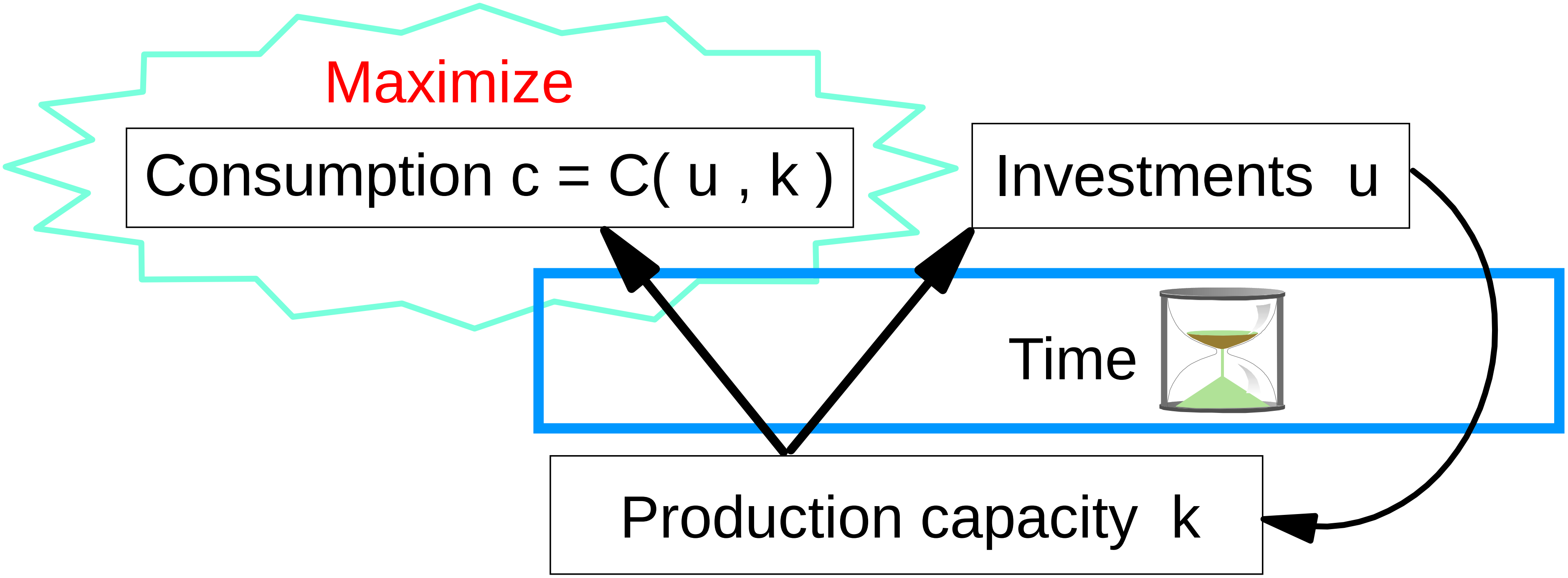
Optimal Control

Consider a simple, closed economy.



- Production capacity $k(t)$ split between consumption $c(t)$ and investments $u(t)$.

Optimal Control with delays



- Want to maximize the total welfare

$$\int_0^\infty e^{-\rho t} \ln C\big(\underbrace{\dot{k}(t) + gk(t-\tau)}_{u(t-\tau)}, k(t-\tau)\big)\, dt.$$

- Parameter ρ relates appreciation of future welfare to present welfare.
- Parameter g describes natural decay of machines, factories etc.

Optimal Control: MFDE

Periodic solutions are interesting from economic point of view.

(1968) Hughes: solving optimal control problem with delays leads to a MFDE.

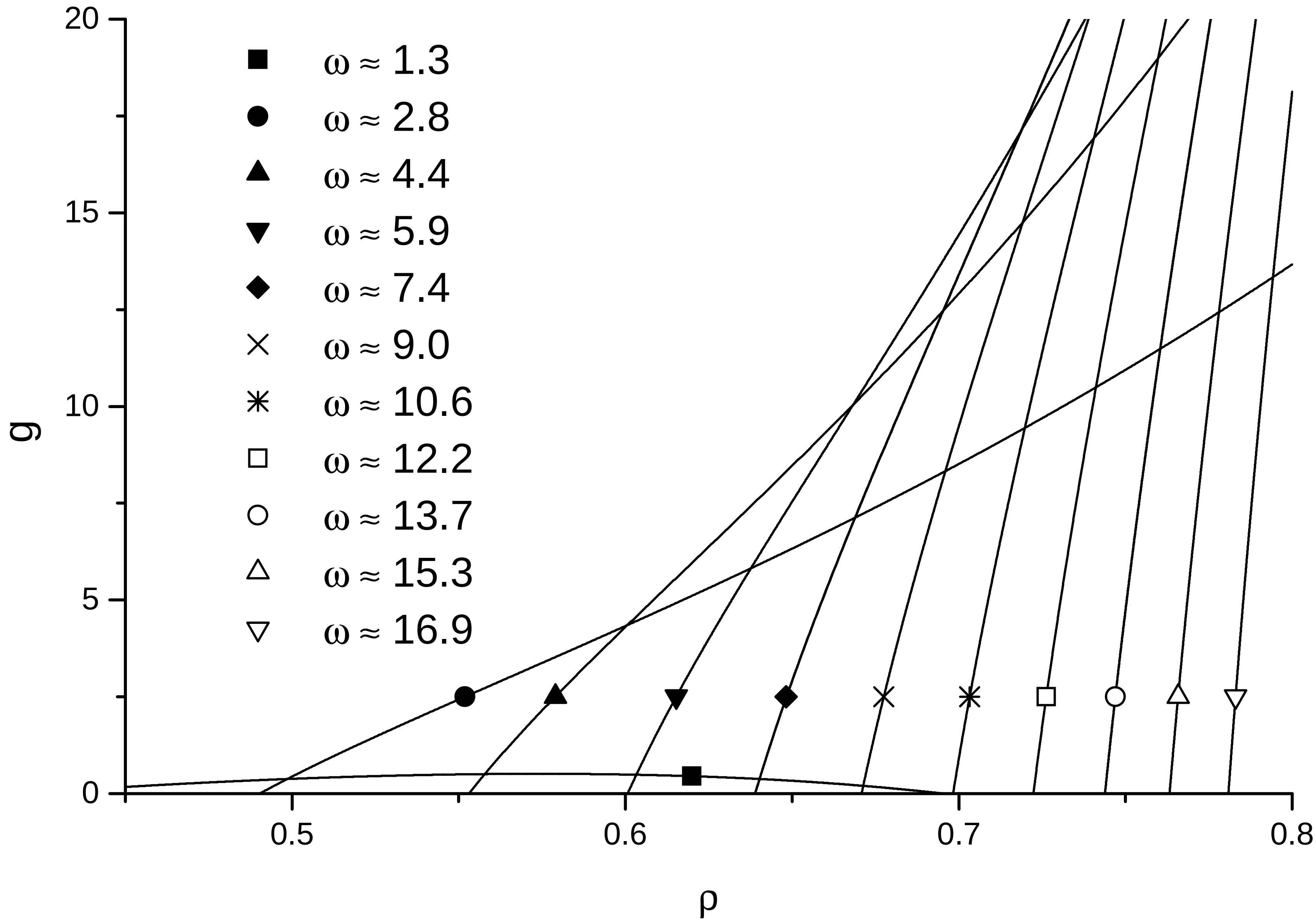
(1979) Benhabib & Nishimura: periodic solutions for model with $\tau = 0$ (no delay).
However, need at least two different types of "machines" (capital goods).

(1989) Rustichini: "formal" analysis of models with $\tau \neq 0$.
However, no rigorous results possible.

(2006) H+VL: Center manifold construction, allowing reduction $\mathrm{MFDE} \to \mathrm{ODE}$.

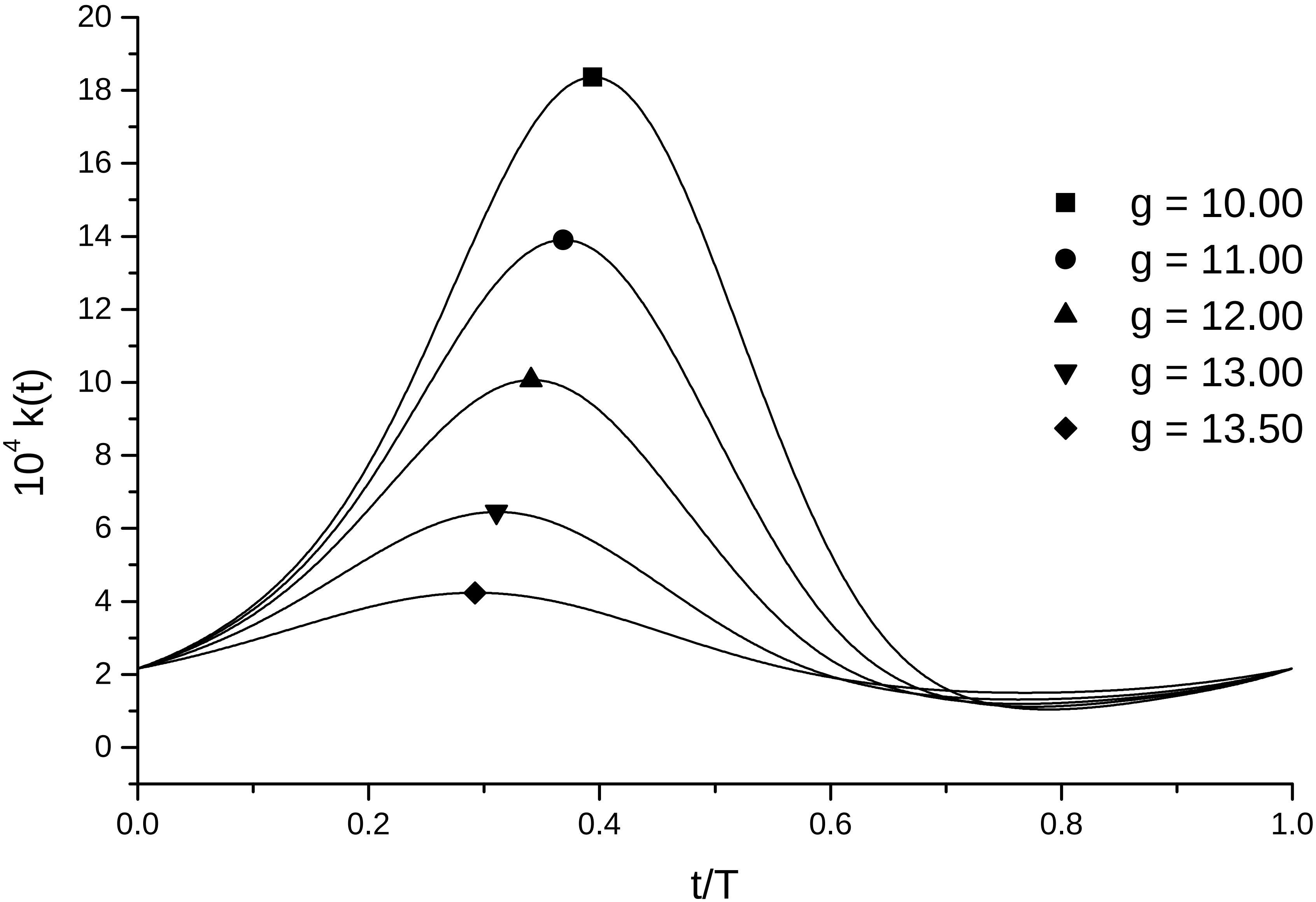
- Allows use of Hopf bifurcation theorem to establish periodic solutions.
- Provides explicit conditions for Hopf bifurcations to occur in terms of parameters ρ and g.
- Need to look for parameters (ρ, g) such that the linear part of the MFDE admits periodic solutions.

Optimal control: Hopf bifurcations



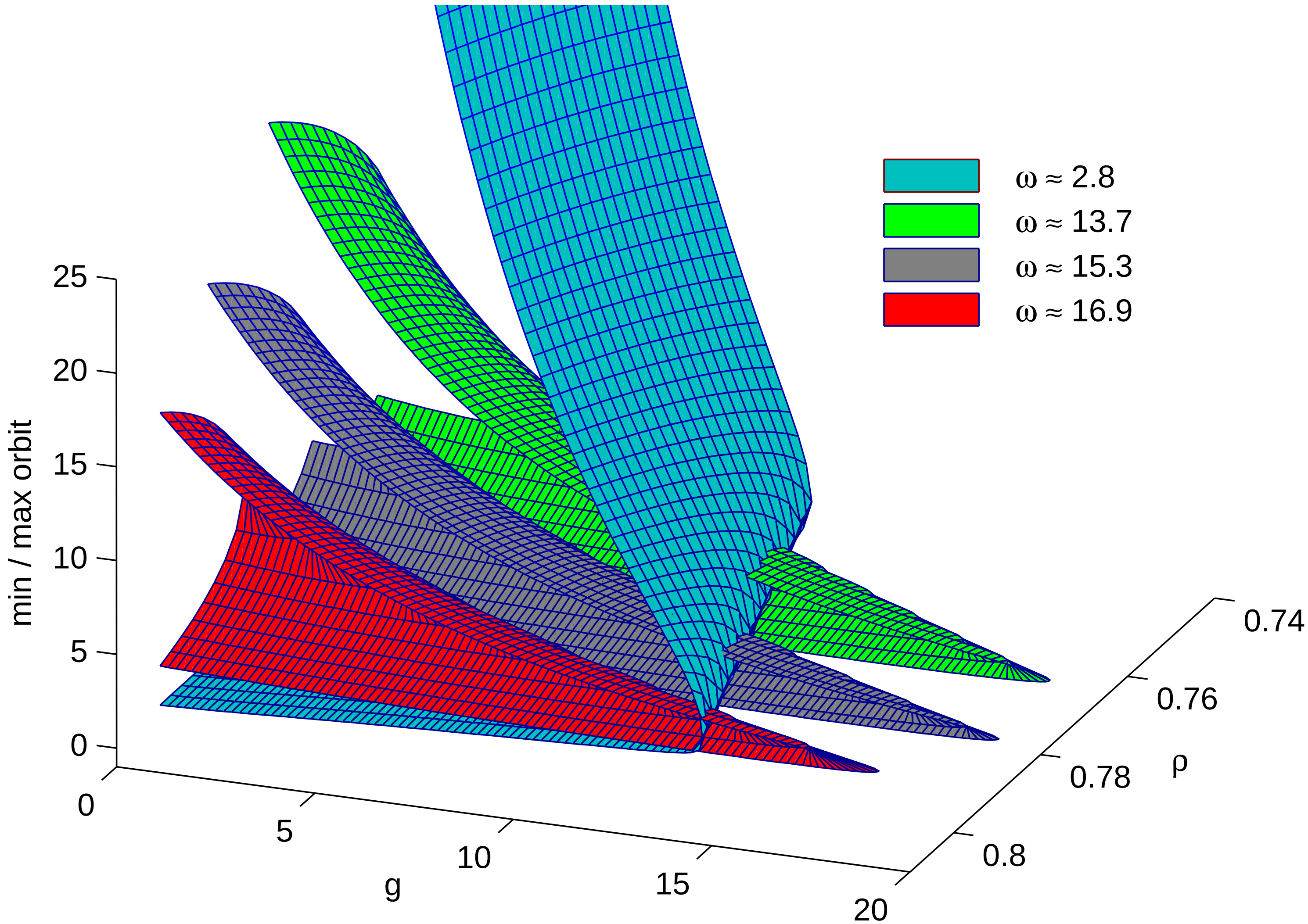
Curves in parameter space where Hopf bifurcations occur.

Optimal control: Periodic Orbits



Examples of periodic orbits for $k(t)$ at different parameter values.

Optimal control: Periodic Orbits



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