5

Distributions

5.1 Test Functions and Distributions

5.1.1 Motivation

Many problems arising naturally in differential equations call for a generalized definition of functions, derivatives, convergence, integrals, etc. In this subsection, we discuss a number of such questions, which will be adequately answered below.

1. In Chapter 1, we noted that any twice differentiable function of the form $u(x,t) = F(x + t) + G(x - t)$ is a solution of the wave equation $u_{tt} = u_{xx}$. Clearly, it seems natural to call $u$ a “generalized” solution even if $F$ and $G$ are not twice differentiable. A natural question is what meaning can be given to $u_{tt}$ and $u_{xx}$ in this case; obviously, they cannot be “functions” in the usual sense. The same question arises for the shock solutions of hyperbolic conservation laws which we discussed in Chapter 3.

2. Consider the ODE initial-value problem

$$u'(t) = f_\epsilon(t), \ u(0) = 0, \quad (5.1)$$

where

$$f_\epsilon(t) = \begin{cases} 1/\epsilon & 1 < t < 1 + \epsilon \\ 0 & \text{otherwise.} \end{cases} \quad (5.2)$$
Obviously, the solution is
\[
 u(t) = \begin{cases} 
 0 & 0 \leq t \leq 1 \\
 (t - 1)/\epsilon & 1 \leq t \leq 1 + \epsilon \\
 1 & t \geq 1 + \epsilon.
\end{cases}
\] (5.3)

Note that the limit of \( u \) as \( \epsilon \to 0 \) exists; it is a step function. The function \( f_\epsilon \) has unit integral; it is supported on shorter and shorter time intervals as \( \epsilon \) tends to zero. It would be natural to regard the "limit" of \( f_\epsilon \) as an instantaneous unit impulse. The question arises what meaning can be given to this limit and in what sense the differential equation holds in the limit. Similar questions arise in many physical problems involving idealized point singularities: the electric field of a point charge, light emitted by a point source, etc.

3. In Chapter 1, we outlined the solution of Dirichlet’s problem by minimizing the integral \( \int_{\Omega} |\nabla u|^2 \, dx \). A fundamental ingredient in turning these ideas into a rigorous theory is obviously the definition of a class of functions for which the integral is finite; the square root of the integral naturally defines a norm on this space of functions. It turns out that \( C^1(\Omega) \) is too restrictive; it is not a complete metric space in the norm defined by the integral. It is natural to consider the completion; this leads to functions for which \( \nabla u \) does not exist in the sense of the classical definition as a pointwise limit of difference quotients.

4. The Fourier transform is a natural tool for dealing with PDEs with constant coefficients posed on all of space. However, the class of functions for which the Fourier integral exists in the conventional sense is rather restrictive; in particular, such functions must be integrable at infinity. Clearly, it would be useful to have a notion of the Fourier transform for functions which do not satisfy such a restriction, e.g., constant functions.

The idea behind generalized functions is roughly this: Given a continuous function \( f(x) \) on \( \Omega \), we can define a linear mapping
\[
 \phi \mapsto \int_{\Omega} f(x)\phi(x) \, dx
\] (5.4)
from a suitable class of functions (which will be called test functions) into \( \mathbb{R} \). We shall see that this mapping has certain continuity properties. A generalized function is then defined to be a linear mapping on the test functions with these same continuity properties.

Since we intend to use generalized functions to study differential equations, a key question is: how do we define derivatives of such functions? The answer is: by using integration by parts. Test functions will be required to
vanish near $\partial \Omega$, so the derivative $\partial f/\partial x_j$ can be defined as the mapping

$$\phi \mapsto -\int_{\Omega} f(x) \frac{\partial \phi}{\partial x_j}(x) \, dx.$$  

(5.5)

Clearly, this definition requires no differentiability of $f$ in the usual sense; the only differentiability requirement is on $\phi$. We shall therefore choose the test functions to be functions with very nice smoothness properties.

### 5.1.2 Test Functions

Let $\Omega$ be a nonempty open set in $\mathbb{R}^m$. We make the following definition.

**Definition 5.1.** A function $f$ defined on $\Omega$ is called a **test function** if $f \in C^\infty(\Omega)$ and there is a compact set $K \subset \Omega$ such that the support of $f$ lies in $K$. The set of all test functions on $\Omega$ is denoted by $\mathcal{D}(\Omega) = C^\infty_0(\Omega)$.

Obviously, $\mathcal{D}(\Omega)$ is a linear space. To do analysis, we need a notion of convergence. It is possible to define open sets in $\mathcal{D}(\Omega)$ and use the notions of general topology. However, for most purposes in PDEs this is not necessary; only a definition for the convergence of sequences is required. This definition is as follows.

**Definition 5.2.** Let $\phi_n, n \in \mathbb{N}$ and $\phi$ be elements of $\mathcal{D}(\Omega)$. We say that $\phi_n$ **converges** to $\phi$ in $\mathcal{D}(\Omega)$, if there is a compact subset $K$ of $\Omega$ such that the supports of all the $\phi_n$ (and of $\phi$) lie in $K$ and, moreover, $\phi_n$ and derivatives of $\phi_n$ of arbitrary order converge uniformly to those of $\phi$.

**Remark 5.3.** Note that the notion of convergence defined above does not come from a metric or norm.

It is often important to know that test functions with certain properties exist; for example one often needs a function that is positive in a small neighborhood of a given point $y$ and zero outside that neighborhood. Such a function can be given explicitly:

$$\phi_{y,\epsilon}(x) = \begin{cases} \exp \left( -\frac{\epsilon^2}{\epsilon^2 - |x-y|^2} \right), & |x - y| < \epsilon \\ 0, & \text{otherwise} \end{cases}$$  

(5.6)

Indeed, this example can be used generate other examples of test functions. The following theorem states that any continuous function of compact support can be approximated uniformly by test functions.

**Theorem 5.4.** Let $K$ be a compact subset of $\Omega$ and let $f \in C(\Omega)$ have support contained in $K$. For $\epsilon > 0$, let

$$f_\epsilon(x) = \frac{1}{C(\epsilon)} \int_K \phi_{y,\epsilon}(x)f(y) \, dy,$$  

(5.7)
where
\[ C(\epsilon) = \int_{\mathbb{R}^m} \phi_{y,\epsilon}(x) \, dy. \] (5.8)

If \( \epsilon < \text{dist}(K, \partial \Omega) \), then \( f_\epsilon \in \mathcal{D}(\Omega) \); moreover, \( f_\epsilon \to f \) uniformly as \( \epsilon \to 0 \).

The proof is left as an exercise.

In a similar fashion, we can construct test functions which are equal to 1 on a given set and equal to 0 on another.

**Theorem 5.5.** Let \( K \) be a compact subset of \( \Omega \) and let \( U \subset \Omega \) be an open set containing \( K \). Then there is a test function which is equal to 1 on \( K \), is equal to 0 outside \( U \) and assumes values in \([0, 1]\) on \( U \setminus K \).

**Proof.** Let \( \epsilon > 0 \) be such that the \( \epsilon \)-neighborhood of \( K \) is contained in \( U \). Let \( K_1 \) be the closure of the \( \epsilon/3 \)-neighborhood of \( K \) and define
\[ f(x) = 1 - \min \left\{ 1, \frac{3}{\epsilon} \text{dist}(x, K_1) \right\}. \] (5.9)

The function \( f \) is continuous, equal to 1 on \( K_1 \) and equal to zero outside of the \( 2\epsilon/3 \)-neighborhood of \( K \). A function with the properties desired by the theorem is given by \( f_{\epsilon/3} \) as defined by (5.7).

Many proofs in PDEs involve a reduction to local considerations in a small neighborhood of a point. (See, for example, Chapter 9.) The device by which this is achieved is known as a partition of unity.

**Definition 5.6.** Let \( U_i, i \in \mathbb{N} \) be a family of bounded open subsets of \( \Omega \) such that

1. the closure of each \( U_i \) is contained in \( \Omega \),
2. every compact subset of \( \Omega \) intersects only a finite number of the \( U_i \) (this property is called local finiteness), and
3. \( \bigcup_{i \in \mathbb{N}} U_i = \Omega \).

A **partition of unity** subordinate to the covering \( \{U_i\} \) is a set of test functions \( \phi_i \) such that

1. \( 0 \leq \phi_i \leq 1 \),
2. \( \text{supp} \phi_i \subset U_i \),
3. \( \sum_{i \in \mathbb{N}} \phi_i(x) = 1 \) for every \( x \in \Omega \).

The following theorem says that such partitions of unity exist.

**Theorem 5.7.** Let \( U_i, i \in \mathbb{N} \) be a collection of sets with the properties stated in Definition 5.6. Then there is a partition of unity subordinate to the covering \( \{U_i\} \).
Proof. We first construct a new covering \( \{V_i\} \), where the \( V_i \) have all the properties of Definition 5.6 and the closure of \( V_i \) is contained in \( U_i \). The \( V_i \) are constructed inductively. Suppose \( V_1, V_2, \ldots, V_{k-1} \) have already been found such that \( U_j \) contains \( V_j \) and

\[
\Omega = \bigcup_{j=1}^{k-1} V_j \cup \bigcup_{j=k}^{\infty} U_j. \tag{5.10}
\]

Let \( F_k \) be the complement of the set

\[
\bigcup_{j=1}^{k-1} V_j \cup \bigcup_{j=k+1}^{\infty} U_j. \tag{5.11}
\]

Then \( F_k \) is a closed set contained in \( U_k \). We choose \( V_k \) to be any open set containing \( F_k \) such that \( \overline{V}_k \subset U_k \). Each point \( x \in \Omega \) is contained in only finitely many of the \( U_i \); hence there is \( N \in \mathbb{N} \) with \( x \notin \bigcup_{j=N+1}^{\infty} U_j \). But this implies that \( x \in \bigcup_{j=1}^{N} V_j \). Hence the \( V_i \) have property 3 of Definition 5.6; the other two properties follow trivially from the fact that \( V_i \subset U_i \).

Let \( W_k \) be an open set such that \( V_k \subset W_k \), \( \overline{W}_k \subset U_k \). According to Theorem 5.5, there is now a test function \( \psi_k \), which is equal to 1 on \( \overline{V}_k \), is equal to zero outside \( W_k \) and takes values between 0 and 1 otherwise. Let

\[
\psi(x) = \sum_{k \in \mathbb{N}} \psi_k(x). \tag{5.12}
\]

Because of property 2 in Definition 5.6, the right-hand side of (5.12) has only finitely many nonzero terms in the neighborhood of any given \( x \), and there is no issue of convergence. The functions \( \phi_k := \psi_k/\psi \) yield the desired partition of unity.

5.1.3 Distributions

We now define the space of distributions. As we indicated in the introduction, the definition of a distribution is constructed very cleverly to achieve two seemingly contradictory goals. We wish to have a generalized notion of a “function” that includes objects that are highly singular or “rough.” At the same time we wish to be able to define “derivatives” of arbitrary order of these objects.

Definition 5.8. A distribution or generalized function is a linear mapping \( \phi \mapsto (f, \phi) \) from \( \mathcal{D}(\Omega) \) to \( \mathbb{R} \), which is continuous in the following sense: If \( \phi_n \to \phi \) in \( \mathcal{D}(\Omega) \), then \( (f, \phi_n) \to (f, \phi) \). The set of all distributions is called \( \mathcal{D}'(\Omega) \).
Example 5.9. Any continuous function \( f \) on \( \Omega \) can be identified with a generalized function by setting

\[
(f, \phi) = \int_{\Omega} f(x)\phi(x) \, dx.
\]  

(5.13)

The continuity of the mapping follows from the familiar theorem concerning the limit of the integral of a uniformly convergent sequence of functions. Indeed, the Lebesgue dominated convergence theorem allows us to make the same claim if \( f \) is merely locally integrable.

Example 5.10. Of course, there are many generalized functions which do not correspond to “functions” in the ordinary sense. The most important example is known as the Dirac delta function. We assume that \( \Omega \) contains the origin, and we define

\[
(\delta, \phi) = \phi(0).
\]  

(5.14)

The continuity of the functional follow from the fact that convergence of a sequence of test functions implies pointwise convergence.

It is easy to show that there is no continuous function satisfying (5.14), (cf. Problem 5.5).

Remark 5.11. Generalized functions like the delta function do not take “values” like ordinary functions. Nevertheless, it is customary to use the language of ordinary functions and speak of “the generalized function \( \delta(x) \).”\(^1\) even though it does not make sense to plug in a specific \( x \). We shall also write \( \int_{\Omega} \delta(x)\phi(x) \, dx \) for \( (\delta, \phi) \).

Example 5.12. For any multiindex \( \alpha \), the mapping

\[
\phi \mapsto D^\alpha \phi(0)
\]

is a generalized function.

Example 5.13. Other singular distributions include such examples from physics as surface charge. If \( S \) is a smooth two-dimensional surface in \( \mathbb{R}^3 \) and \( q : S \to \mathbb{R} \) is integrable, then for \( \phi \in \mathcal{D}(\mathbb{R}^3) \) we define

\[
(q, \phi) = \int_S q(x)\phi(x) \, da(x)
\]

where \( da(x) \) indicates integration with respect to surface area on \( S \).

Example 5.14. A current flowing along a curve \( C \subset \mathbb{R}^3 \) is an example of a vector-valued distribution. If \( j : C \to \mathbb{R}^3 \) is integrable, then for \( \phi \in \mathcal{D}(\mathbb{R}^3)^3 \) we define

\[
(j, \phi) = \int_C j(x) \cdot \phi(x) \, d\sigma(x)
\]

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\(^1\)We apologize to those among our friends to whom such language is an abomination — even for ordinary functions!
where \( d\sigma(x) \) indicates integration with respect to arclength on \( C \).

**Remark 5.15.** Of course, complex-valued distributions can be defined in the same fashion as real-valued distributions; in that case, however, it is customary to make the convention

\[
(f, \phi) = \int_{\Omega} f(x)\phi(x)\,dx
\]  

for every \( \phi \in \mathcal{D}(\Omega) \) with support contained in \( K \).

**Lemma 5.16.** Let \( f \in \mathcal{D}'(\Omega) \) and let \( K \) be a compact subset of \( \Omega \). Then there exists \( n \in \mathbb{N} \) and a constant \( C \) such that

\[
|\langle f, \phi \rangle| \leq C \sum_{|\alpha| \leq n} \max_{x \in K} |D^\alpha \phi(x)|
\]  

for every \( \phi \in \mathcal{D}(\Omega) \) with support contained in \( K \).

**Proof.** Suppose not. Then for every \( n \) there exists \( \psi_n \) such that

\[
|\langle f, \psi_n \rangle| > n \sum_{|\alpha| \leq n} \max_{x \in K} |D^\alpha \psi_n(x)|.
\]

Let \( \phi_n := \psi_n/|\langle f, \psi_n \rangle| \). Then \( \phi_n \to 0 \) in \( \mathcal{D}(\Omega) \), but \( \langle f, \phi_n \rangle \equiv 1 \). This is a contradiction, and the proof is complete. \( \square \)

We conclude this subsection with some straightforward definitions.

**Definition 5.17.** For distributions \( f \) and \( g \) and real number \( \alpha \in \mathbb{R} \), we set

\[
\langle f + g, \phi \rangle = \langle f, \phi \rangle + \langle g, \phi \rangle,
\]

\[
\langle \alpha f, \phi \rangle = \langle f, \alpha \phi \rangle.
\]

(If \( \alpha \) is allowed to be complex, then the right-hand side of (5.18) is changed to \( \langle f, \overline{\alpha} \phi \rangle \).)

**Remark 5.18.** It is in general not possible to define the product of two generalized functions (cf. Problems 5.11, 5.12). However, we can define the product of a distribution and a smooth function.

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\(^2\)One of the oldest problems in Hilbert space theory is whether to put the complex conjugate on the first or on the second factor in the inner product. The convention made here is widely followed by physicists. Pure mathematicians tend to make the opposite convention.
**Definition 5.19.** For any function \( a \in C^\infty(\Omega) \), we define

\[
(af, \phi) = (f, a\phi) \tag{5.19}
\]

If the graph of a function \( f(x) \) is shifted by \( h \), one obtains the graph of the function \( f(x - h) \), i.e., \( x \) is shifted by \(-h\). This can be generalized to distributions on \( \mathbb{R}^m \).

**Definition 5.20.** Let \( U(x) = Ax + b \) be a nonsingular linear transformation in \( \mathbb{R}^m \), and let \( U^{-1}(y) = A^{-1}(y - b) \) be the inverse transformation. Then we set

\[
(Uf, \phi) = |\text{det} A| (f(x), \phi(U(x))). \tag{5.20}
\]

This definition is motivated by the following formal calculation:

\[
(Uf, \phi) = (f(U^{-1}(x)), \phi(x)) = \int_{\mathbb{R}^m} f(U^{-1}(x))\phi(x) \, dx = |\text{det} A| \int_{\mathbb{R}^m} f(y)\phi(U(y)) \, dy.
\]

(We have substituted \( x = U(y) \).)

**Example 5.21.** The translation \( \delta(x - x_0) \) is defined as

\[
(\delta(x - x_0), \phi(x)) = (\delta(x), \phi(x + x_0)) = \phi(x_0). \tag{5.21}
\]

**Remark 5.22.** With this definition, we can define the symmetry of a generalized function; for example, \( f \) is even if \( f(-x) = f(x) \), i.e.,

\[
(f(x), \phi(x)) = (f(x), \phi(-x)).
\]

### 5.1.4 Localization and Regularization

Although generalized functions cannot be evaluated at points, they can be restricted to open sets. This is quite straightforward. If \( G \) is an open subset of \( \Omega \), then \( D(G) \) is naturally embedded in \( D(\Omega) \), and hence every generalized function on \( \Omega \) defines a generalized function on \( G \) by restriction. Consequently, we shall define the following.

**Definition 5.23.** We say that \( f \in D'(\Omega) \) **vanishes** on and open set \( G \subset \Omega \) if \( (f, \phi) = 0 \) for every \( \phi \in D(G) \). Two distributions are **equal** on \( G \) if their difference vanishes on \( G \).

It can be shown (cf. Problem 5.7) that if \( f \) vanishes locally near every point of \( G \), i.e., if every point of \( G \) has a neighborhood on which \( f \) vanishes, then \( f \) vanishes on \( G \). An immediate consequence is that if \( f \) vanishes on each of a family of open sets, it also vanishes on their union. Hence there is a largest open set \( N_f \) on which \( f \) vanishes.

**Definition 5.24.** The complement of \( N_f \) in \( \Omega \) is called the **support** of \( f \).
Example 5.25. The support of the delta function is the set \{0\}. Although the delta function cannot be evaluated at points, it makes sense to say that it vanishes except at the origin.

Remark 5.26. Functions with nonintegrable singularities are not defined as generalized functions by equation (5.13). However, it is often possible to define a generalized function which agrees with a singular function on any open set that does not contain the singularity. Such a generalized function is called a regularization. For example, a regularization of the function $1/x$ on $\mathbb{R}$ is given by the principal value integral

$$ (f, \phi) = \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} \, dx + \int_{-\epsilon}^{\epsilon} \frac{\phi(x) - \phi(0)}{x} \, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x} \, dx \quad (5.22) $$

(cf. Problem 5.9).

5.1.5 Convergence of Distributions

Just as sequences of classical functions are central to PDEs, so are sequences of generalized functions.

Definition 5.27. A sequence $f_n$ in $\mathcal{D}'(\Omega)$ converges to $f \in \mathcal{D}'(\Omega)$ if

$$(f_n, \phi) \to (f, \phi)$$

for every $\phi \in \mathcal{D}(\Omega)$.

Example 5.28. A uniformly convergent sequence of continuous functions (which define distributions as in Example 5.9) also converges in $\mathcal{D}'$.

Example 5.29. Consider the sequence

$$ f_n(x) = \begin{cases} n, & 0 < x < 1/n \\ 0, & \text{otherwise} \end{cases} \quad (5.23) $$

We have

$$ \int_{-\infty}^{\infty} f_n(x) \phi(x) \, dx = n \int_{0}^{1/n} \phi(x) \, dx, \quad (5.24) $$

which converges to $\phi(0)$ as $n \to \infty$. Hence $f_n(x) \to \delta(x)$ in $\mathcal{D}'(\mathbb{R})$.

Remark 5.30. Problem 5.10 asks the reader to prove that every distribution is the limit of distributions with compact support. Later we shall actually see that every distribution is a limit of test functions; in other words, test functions are dense in $\mathcal{D}'(\Omega)$.

Another important result is the (sequential) completeness of $\mathcal{D}'(\Omega)$.

Theorem 5.31. Let $f_n$ be a sequence in $\mathcal{D}'(\Omega)$ such that $(f_n, \phi)$ converges for every $\phi \in \mathcal{D}(\Omega)$. Then there exists $f \in \mathcal{D}'(\Omega)$ such that $f_n \to f$. 
Proof. We define
\[(f, \phi) = \lim_{n \to \infty} (f_n, \phi).\] (5.25)

Obviously, \(f\) is a linear mapping from \(D(\Omega)\) to \(\mathbb{R}\). To verify that \(f \in D'(\Omega)\), we have to establish its continuity, i.e., we must show that if \(\phi_n \to 0\) in \(D(\Omega)\), then \((f, \phi_n) \to 0\). Assume the contrary. Then, after choosing a subsequence which we again label \(\phi_n\), we may assume \(\phi_n \to 0\), but \(|(f, \phi_n)| \geq c > 0\).

Now recall that convergence to 0 in \(D(\Omega)\) means that the supports of all the \(\phi_n\) lie in a fixed compact subset of \(\Omega\) and that all derivatives of the \(\phi_n\) converge to zero uniformly. After again choosing a subsequence, we may assume that \(|D^\alpha \phi_n(x)| \leq 4^{-n}\) for \(|\alpha| \leq n\). Let now \(\psi_n = 2^n \phi_n\). Then the \(\psi_n\) still converge to 0 in \(D(\Omega)\), but \(|(f, \psi_n)| \to \infty\).

We shall now recursively construct a subsequence \(\{f'_n\}\) of \(\{f_n\}\) and a subsequence \(\{\psi'_n\}\) of \(\{\psi_n\}\). First we choose \(\psi'_1\) such that \(|(f, \psi'_1)| > 1\). Since \((f_n, \psi'_1) \to (f, \psi'_1)\), we may choose \(f'_1\) such that \(|(f'_1, \psi'_1)| > 1\). Now suppose we have chosen \(f'_j\) and \(\psi'_j\) for \(j < n\). We then choose \(\psi'_n\) from the sequence \(\{\psi_n\}\) such that
\[|(f'_j, \psi'_n)| < \frac{1}{2^{n-j}}, \quad j = 1, 2, \ldots, n - 1,\] (5.26)
\[|(f, \psi'_n)| > \sum_{j=1}^{n-1} |(f, \psi'_j)| + n.\] (5.27)

This is possible because, on the one hand, \(\psi_n \to 0\), and, on the other hand, \(|(f, \psi_n)| \to \infty\). Since, moreover, \((f_n, \psi) \to (f, \psi)\), we can choose \(f'_n\) such that
\[|(f'_n, \psi'_n)| > \sum_{j=1}^{n-1} |(f'_n, \psi'_j)| + n.\] (5.28)

Next we set
\[\psi = \sum_{n=1}^{\infty} \psi'_n.\] (5.29)

\(^3\)The use of the same symbol for both a sequence and any of its subsequences is a typical practice in PDEs. Its primary purpose is clarity of notation (since we often have to consider subsequences several levels deep), but it has the pleasant side effect of driving many classical analysts crazy. Of course, there are cases where it is important to distinguish between a sequence and its subsequence (as we do later in this proof) and we do so with appropriate notation.
It follows from the construction of the $\psi'_n$ that the series on the right converges in $\mathcal{D}(\Omega)$. Hence

$$\langle f'_n, \psi \rangle = \sum_{j=1}^{n-1} \langle f'_n, \psi'_j \rangle + \langle f'_n, \psi'_n \rangle + \sum_{j=n+1}^{\infty} \langle f'_n, \psi'_j \rangle. \quad (5.30)$$

From (5.26) we find that

$$\left| \sum_{j=n+1}^{\infty} \langle f'_n, \psi'_j \rangle \right| < \sum_{j=n+1}^{\infty} 2^{n-j} = 1, \quad (5.31)$$

and this in conjunction with (5.30) and (5.28) implies that $|\langle f'_n, \psi \rangle| > n - 1$. Hence the limit of $\langle f'_n, \psi \rangle$ as $n \to \infty$ does not exist, a contradiction. \(\square\)

A similar contradiction argument can be used to prove the following lemma; the details of the proof are left as an exercise (cf. Problem 5.18).

**Lemma 5.32.** Assume that $f_n \to 0$ in $\mathcal{D}'(\Omega)$ and $\phi_n \to 0$ in $\mathcal{D}(\Omega)$. Then $(f_n, \phi_n) \to 0$.

We also have the following corollary.

**Corollary 5.33.** If $f_n \to f$ and $\phi_n \to \phi$, then $(f_n, \phi_n) \to (f, \phi)$.

Hence the pairing between distributions and test functions is continuous. (Of course separate continuity in each factor is obvious from the definitions, but joint continuity requires a proof.)

**Proof.** The corollary follows immediately from the identity

$$\langle f_n - f, \phi_n - \phi \rangle = \langle f_n, \phi_n \rangle - \langle f, \phi_n \rangle - \langle f_n, \phi \rangle + \langle f, \phi \rangle. \quad (5.32)$$

\(\square\)

### 5.1.6 Tempered Distributions

It is possible to define different spaces of test functions and, correspondingly, of distributions. In particular, for $\Omega = \mathbb{R}^m$, it is natural to replace the requirement of compact support by one of rapid decay at infinity. This leads to the following definition.

**Definition 5.34.** Let $\mathcal{S}(\mathbb{R}^m)$ be the space of all complex-valued functions on $\mathbb{R}^m$ which are of class $C^\infty$ and such that $|x|^k |D^\alpha \phi(x)|$ is bounded for every $k \in \mathbb{N}$ and every multi-index $\alpha$. We say that a sequence $\phi_n$ in $\mathcal{S}(\mathbb{R}^m)$ converges to $\phi$ if the derivatives of all orders of the $\phi_n$ converge uniformly to those of $\phi$ and the constants $C_{k\alpha}$ in the bounds $|x|^k |D^\alpha \phi_n(x)| \leq C_{k\alpha}$ can be chosen independently of $n$.

Obviously, $\mathcal{D}(\mathbb{R}^m)$ is a subspace of $\mathcal{S}(\mathbb{R}^m)$. Moreover, $\mathcal{D}(\mathbb{R}^m)$ is dense in $\mathcal{S}(\mathbb{R}^m)$. To see this, let $e(x)$ be a $C^\infty$-function which is equal to 1 in the
unit ball and vanishes outside the ball of radius 2. Let \( e_n(x) = e(x/n) \).

Then, for any \( f \in \mathcal{S}(\mathbb{R}^m) \), we have \( f = \lim_{n \to \infty} f e_n \).

We now define the tempered distributions to be continuous linear functionals on \( \mathcal{S} \).

**Definition 5.35.** A **tempered distribution** on \( \mathbb{R}^m \) is a linear mapping \( \phi \mapsto (f, \phi) \) from \( \mathcal{S}(\mathbb{R}^m) \) to \( \mathbb{C} \) with the continuity property that \( (f, \phi_n) \to (f, \phi) \) if \( \phi_n \to \phi \) in \( \mathcal{S}(\mathbb{R}^m) \). The set of all tempered distributions is denoted by \( \mathcal{S}'(\mathbb{R}^m) \). We say that \( f_n \to f \) in \( \mathcal{S}'(\mathbb{R}^m) \) if \( (f_n, \phi) \to (f, \phi) \) for every \( \phi \in \mathcal{S}(\mathbb{R}^m) \).

Clearly, every tempered distribution defines a distribution by restriction. Moreover, if two tempered distributions agree as elements of \( \mathcal{D}'(\mathbb{R}^m) \), they also agree as elements of \( \mathcal{S}'(\mathbb{R}^m) \); this follows from the fact that \( \mathcal{D}(\mathbb{R}^m) \) is dense in \( \mathcal{S}(\mathbb{R}^m) \). Hence \( \mathcal{S}'(\mathbb{R}^m) \) is a linear subspace of \( \mathcal{D}'(\mathbb{R}^m) \). Moreover, convergence in \( \mathcal{S}'(\mathbb{R}^m) \) obviously implies convergence in \( \mathcal{D}'(\mathbb{R}^m) \).

**Problems**

5.1. Show that \( \phi_{\mathbf{y}, \epsilon} \in \mathcal{D}(\mathbb{R}^m) \).

5.2. Show that the sequence \( \phi_n(x) = n^{-1} \phi_{0, \epsilon}(x) \) converges to zero in \( \mathcal{D}(\mathbb{R}^m) \). Show that the sequence \( \psi_n(x) = n^{-1} \phi_{0, \epsilon}(x/n) \) converges to zero uniformly and so do all derivatives. Why does \( \psi_n \) nevertheless not converge to zero in \( \mathcal{D}(\mathbb{R}^m) \)? Does it converge to zero in \( \mathcal{S}(\mathbb{R}^m) \)?

5.3. Prove Theorem 5.4.

5.4. Let \( f \) and \( g \) be two different functions in \( C(\Omega) \). Show that they also differ as generalized functions.

5.5. Show that the Dirac delta function cannot be identified with any continuous function.

5.6. Explain what it means for a generalized function to be periodic or radially symmetric.

5.7. Let \( f \) be a generalized function on \( \Omega \) and let \( G \) be an open subset of \( \Omega \). Assume that every point in \( G \) has a neighborhood on which \( f \) vanishes. Prove that \( f \) vanishes on \( G \). Hint: Use a partition of unity argument.

5.8. Prove that if \( \phi \) vanishes in a neighborhood of the support of \( f \), then \( (f, \phi) = 0 \). Would it suffice if \( \phi \) vanishes on the support of \( f \)?

5.9. Show that (5.22) does indeed define a generalized function and that the definition does not depend on \( \epsilon \). How can one define a regularization of \( 1/x^2 \)?

5.10. Prove that every distribution is the limit of a sequence of distributions with compact support. Hint: Let \( f_n = f \psi_n \), where \( \psi_n \) is a \( C^\infty \) cutoff function.
5.11. Show that
\[ \lim_{n \to \infty} \sin(nx) = 0 \]
in \( \mathcal{D}'(\mathbb{R}) \), but that
\[ \lim_{n \to \infty} \sin^2(nx) \neq 0. \]
Hence multiplication of distributions is not a continuous operation even where it is defined.

5.12. Let \( f_n \) be the sequence defined by (5.23). Show that
\[ \lim_{n \to \infty} f_n^2 \]
does not exist in the sense of distributions. Show, however, that
\[ \lim_{n \to \infty} f_n^2 - n\delta \]
exists.

5.13. Find
\[ \lim_{n \to \infty} \sqrt{n} \exp(-nx^2) \]
in the sense of distributions.

5.14. Exhibit a sequence in \( S'(\mathbb{R}) \) which converges to zero in \( \mathcal{D}'(\mathbb{R}) \), but not in \( S'(\mathbb{R}) \).

5.15. Show that the sequence \( \phi_n \) converges in \( S(\mathbb{R}^m) \) if and only if \( |x|^k D^\alpha \phi_n(x) \) converges uniformly for every \( k \in \mathbb{N} \cup \{0\} \) and every \( \alpha \).

5.16. Show that \( S'(\mathbb{R}^m) \) is sequentially complete.

5.17. Let \( U_i, i \in \mathbb{N} \) be open sets such that \( \Omega = \bigcup_{i \in \mathbb{N}} U_i \). Let \( f_i \in \mathcal{D}'(U_i) \) be given such that \( f_i \) and \( f_j \) agree on \( U_i \cap U_j \). Show that there exists \( f \in \mathcal{D}'(\Omega) \) such that \( f = f_i \) on \( U_i \).

5.18. Prove Lemma 5.32.

5.19. (a) Let \( \Omega \) be any open subset of \( \mathbb{R}^m \). Show that a family of subsets with the properties of Definition 5.6 exists.

(b) Let \( \{U_i\} \) be any countable covering of \( \Omega \) by open sets. A refinement of \( \{U_i\} \) is a covering by open sets \( V_k \), where each \( V_k \) is a subset of one of the \( U_i \). Given any covering of \( \Omega \) by open sets, show that there is a refinement satisfying the properties of Definition 5.6.
5.2 Derivatives and Integrals

5.2.1 Basic Definitions

In this section, we discuss differentiation of distributions and various applications. We shall confine our discussion to distributions in $\mathcal{D}'(\Omega)$; completely analogous considerations apply in $\mathcal{S}'(\mathbb{R}^m)$. We define the derivative of a distribution as follows.

**Definition 5.36.** Let $f \in \mathcal{D}'(\Omega)$. Then the derivative of $f$ with respect to $x_j$ is defined as

$$
\left( \frac{\partial f}{\partial x_j}, \phi \right) = - \left( f, \frac{\partial \phi}{\partial x_j} \right). \quad (5.33)
$$

**Remark 5.37.** If $f$ is in $C^1(\Omega)$, this definition agrees with the classical derivative, as can be seen by an integration by parts. It is easy to see that $\partial f/\partial x_j$ is again in $\mathcal{D}'(\Omega)$.

Remark that differentiation is a continuous operation in $\mathcal{D}'(\Omega)$, i.e., the reader can show the following.

**Theorem 5.38.** If $f_n \to f$ in $\mathcal{D}'(\Omega)$ then $\partial f_n/\partial x_j \to \partial f/\partial x_j$.

Thus, for distributions exchanging derivatives and limits is no problem, quite a contrast to the situation in classical calculus.

**Remark 5.39.** Higher derivatives are defined recursively; the equality of mixed partial derivatives is obvious from the definition and the equality of the mixed partial derivatives of test functions. In general, we have

$$
(D^\alpha f, \phi) = (-1)^{|\alpha|}(f, D^\alpha \phi). \quad (5.34)
$$

**Remark 5.40.** The classical derivative is defined as a limit of difference quotients. In a sense, distributional derivatives are still limits of difference quotients. In the previous section, we defined the translation of a distribution by

$$
(f(x + h e_j), \phi(x)) = (f(x), \phi(x - h e_j)). \quad (5.35)
$$

This does not necessarily make sense, because $x - h e_j$ need not lie in $\Omega$. For fixed $\phi \in \mathcal{D}(\Omega)$, however, $\phi(x - h e_j)$ is in $\mathcal{D}(\Omega)$ provided $h$ is sufficiently small. Hence (5.35) is meaningful for small $h$, although how small $h$ has to
be depends on $\phi$. We now find

$$
\lim_{h \to 0} \frac{1}{h} \left[ (f(x + he_j), \phi(x)) - (f(x), \phi(x)) \right]
= \lim_{h \to 0} \left( f(x), \frac{1}{h} (\phi(x - he_j) - \phi(x)) \right)
= -\left( f, \frac{\partial \phi}{\partial x_j} \right)
= \left( \frac{\partial f}{\partial x_j}, \phi \right).
$$

### 5.2.2 Examples

**Example 5.41.** Consider the function

$$
H(x) = \begin{cases} 
0, & x \leq 0 \\
1, & x > 0.
\end{cases} \quad (5.36)
$$

We compute

$$
(H', \phi) = -(H, \phi') = -\int_{0}^{\infty} \phi'(x) \, dx = \phi(0) = (\delta, \phi),
$$

i.e., the derivative of $H$ is the delta function. The function $H$ is called the Heaviside function.

**Example 5.42.** The $k$th derivative of the delta function is the functional $\phi \mapsto (-1)^k \phi^{(k)}(0)$.

**Example 5.43.** Let

$$
x_+^\lambda = \begin{cases} 
0, & x \leq 0 \\
x^\lambda, & x > 0
\end{cases} \quad (5.38)
$$

and $-1 < \lambda < 0$. Naively, one may expect that the derivative is $\lambda x_+^{\lambda-1}$, but this function has a nonintegrable singularity and hence it is not a distribution. The proper answer is an appropriate regularization. We
compute

\[
((x^\lambda)'_+, \phi) = -(x^\lambda_+, \phi') \\
= -\int_0^\infty x^\lambda \phi'(x) \, dx \\
= -\lim_{\varepsilon \to 0} \int_\varepsilon^\infty x^\lambda \phi'(x) \, dx \\
= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \lambda x^{\lambda-1}(\phi(x) - \phi(\varepsilon)) \, dx \\
= \int_0^\infty \lambda x^{\lambda-1}(\phi(x) - \phi(0)) \, dx.
\]

**Example 5.44.** Let \( \Omega \) be a domain with smooth boundary \( \Gamma \). Let \( f \) be in \( C^1(\overline{\Omega}) \) and let \( f = 0 \) in the exterior of \( \Omega \). We regard \( f \) as a distribution on \( \mathbb{R}^m \). We find

\[
\left( \frac{\partial f}{\partial x_j}, \phi \right) = -\left( f, \frac{\partial \phi}{\partial x_j} \right) \\
= -\int_{\Omega} f(x) \frac{\partial \phi}{\partial x_j}(x) \, dx \\
= \int_{\Omega} \frac{\partial f}{\partial x_j}(x) \phi(x) \, dx - \int_{\Gamma} f(x)\phi(x)n_j \, dS.
\]

Here \( n_j \) is the \( j \)th component of the unit outward normal to \( \Gamma \) and \( dS \) is differential \( m-1 \) dimensional surface area. Thus, the distributional derivative of \( f \) involves one term corresponding to the ordinary derivative in \( \Omega \) and another term involving a distribution supported on \( \Gamma \). This latter term results from the jump of \( f \) across \( \Gamma \).

**Example 5.45.** The previous example has some important applications in electromagnetism. Let \( \Omega \subset \mathbb{R}^3 \) be a domain with smooth boundary \( \Gamma \). Suppose we have a polarization vector field \( \mathbf{p} : \overline{\Omega} \to \mathbb{R}^3 \) which is in \( C^1(\overline{\Omega}) \). By setting \( \mathbf{p} = 0 \) in the exterior of \( \Omega \), we can regard \( \mathbf{p} \) as a distribution on \( \mathbb{R}^3 \). We then define the polarization charge to be the divergence of \( \mathbf{p} \) in the sense of distributions. We calculate this as follows.

\[
(\nabla \cdot \mathbf{p}, \phi) = -\sum_{i=1}^3 \left( p_i, \frac{\partial \phi}{\partial x_i} \right) \\
= -\int_{\Omega} \mathbf{p}(x) \cdot \nabla \phi(x) \, dx \\
= \int_{\Omega} \nabla \cdot \mathbf{p}(x)\phi(x) \, dx - \int_{\Gamma} \mathbf{p}(x) \cdot \mathbf{n}(x)\phi(x) \, dA.
\]
Here \( \mathbf{n} \) is the unit outward normal to \( \Gamma \) and \( dA \) is differential surface area on \( \Gamma \). Thus, the polarization charge involves one term corresponding to the ordinary divergence of \( \mathbf{p} \) in \( \Omega \) and surface charge given by the normal component of \( \mathbf{p} \) on \( \Gamma \). This latter term results from the jump of \( \mathbf{p} \) across \( \Gamma \). If \( \mathbf{p} \) was piecewise smooth with surfaces of jump discontinuity in the interior of \( \Omega \), the normal components of the jumps along these surfaces would contribute polarization charge as well.

**Example 5.46.** Let \( f(x) = 1/|x| = 1/r \) on \( \mathbb{R}^3 \). It is easy to check that \( \Delta(1/r) = 0 \) for \( r \neq 0 \). We shall evaluate the Laplacian of \( 1/r \) in the distributional sense. We compute

\[
\left( \Delta \frac{1}{r}, \phi \right) = \left( \frac{1}{r}, \Delta \phi \right) = \int_{\mathbb{R}^3} \frac{\Delta \phi(x)}{r} \, dx = \lim_{\epsilon \to 0} \int_{r \geq \epsilon} \frac{\Delta \phi}{r} \, dx.
\]

Integration by parts yields

\[
\int_{r \geq \epsilon} \frac{\Delta \phi}{r} \, dx = \int_{r \geq \epsilon} \Delta \left( \frac{1}{r} \right) \phi \, dx - \int_{r = \epsilon} \frac{\partial \phi}{\partial r} \frac{1}{r} \, dS + \int_{r = \epsilon} \phi \frac{\partial}{\partial r} \frac{1}{r} \, dS. \quad (5.39)
\]

On the right-hand side of (5.39), the first term vanishes, the second is of order \( \epsilon \) as \( \epsilon \to 0 \) and the last term is equal to \( -\epsilon^{-2} \int_{r = \epsilon} \phi \, dS \), i.e., to \( -4\pi \) times the average of \( \phi \) on the sphere of radius \( \epsilon \). Letting \( \epsilon \to 0 \), we therefore obtain

\[
\left( \Delta \frac{1}{r}, \phi \right) = -4\pi \phi(0), \quad (5.40)
\]

i.e.,

\[
\Delta \frac{1}{r} = -4\pi \delta. \quad (5.41)
\]

Solutions of the equation \( Lu = \delta \), where \( L \) is a partial differential operator with constant coefficients, are of considerable importance; we shall investigate more such solutions in the next two sections.

**Example 5.47.** In this and the following example, we exploit the fact that differentiation is a continuous operation. Let us consider the Fourier series

\[
\cos x + \cos 2x + \cdots + \cos nx + \cdots. \quad (5.42)
\]

Clearly, this series does not converge in the ordinary sense; for example, it diverges for \( x = 0 \). However, the series

\[
\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots \quad (5.43)
\]
converges to \((\pi - x)/2\) on \((0, 2\pi)\), uniformly on every compact subinterval, and the partial sums of the series (5.43) are uniformly bounded on \(\mathbb{R}\). (We shall not prove these claims here; instead we refer to texts on Fourier series or advanced calculus or to the discussion of Fourier series in Chapter 6.) From this, it is clear that (5.43) converges in the sense of distributions to the \(2\pi\)-periodic continuation of \((\pi - x)/2\); that is, a “sawtooth wave” with jumps at integer multiples of \(2\pi\). We can write this in terms of the Heaviside function.

\[
\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots = \frac{\pi - x}{2} + \pi \sum_{n=1}^{\infty} H(x - 2\pi n) - \pi \sum_{n=0}^{\infty} H(-2\pi n - x)
\]

We now obtain (5.42) by differentiation and the symmetry of the dirac delta:

\[
\cos x + \cos 2x + \cos 3x + \cdots = \frac{d}{dx} \left( \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots \right) = -\frac{1}{2} + \pi \sum_{n \in \mathbb{Z}} \delta(x - 2\pi n) \quad (5.44)
\]

(cf. Problem 5.20).

**Example 5.48.** To prove that a sequence of integrable functions \(f_n : \mathbb{R} \to \mathbb{R}\) converges to the delta function, it suffices to show that the primitives converge to the Heaviside function. The following conditions are sufficient for this:

1. For any \(\epsilon > 0\), we have
   \[
   \lim_{n \to \infty} \int_{-\infty}^{-a} f_n(x) \, dx = 0, \quad \lim_{n \to \infty} \int_{a}^{\infty} f_n(x) \, dx = 0 \quad (5.45)
   \]
   uniformly for \(a \in [\epsilon, \infty)\);

2. \[
   \lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x) \, dx = 1; \quad (5.46)
   \]

3. \(|\int_{-\infty}^{a} f_n(x) \, dx|\) is bounded by a constant independent of \(a \in \mathbb{R}\) and \(n \in \mathbb{N}\).

Examples of functions satisfying these conditions are

\[
f_\epsilon(x) = \frac{\epsilon}{\pi(x^2 + \epsilon^2)}, \quad \epsilon \to 0, \quad (5.47)
\]

\[
f_t(x) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad t \to 0^+, \quad (5.48)
\]
\[ f_n(x) = \frac{\sin (nx)}{\pi x}, \quad n \to \infty. \quad (5.49) \]

5.2.3 **Primitives and Ordinary Differential Equations**

If the derivatives of a function vanish, the function is a constant. We shall now establish the analogous result for distributions.

**Theorem 5.49.** Let \( \Omega \) be connected, and let \( u \in \mathcal{D}'(\Omega) \) be such that \( \nabla u = 0 \). Then \( u \) is a constant.

**Proof.** We first consider the one-dimensional case. Let \( \Omega = I \) be an interval. The condition that \( u' = 0 \) means that \( (u, \phi') = 0 \) for every \( \phi \in \mathcal{D}(I) \). In other words, \( (u, \psi) = 0 \) for every test function \( \psi \) which is the derivative of a test function. It is easy to see that \( \psi \) is the derivative of a test function iff \( \int_I \psi(x) \, dx = 0 \). Let now \( \phi_0 \) be any test function with unit integral. Then any \( \phi \in \mathcal{D}(I) \) can be decomposed as

\[ \phi(x) = \phi_0(x) \int_I \phi(s) \, ds + \psi(x), \quad (5.50) \]

where the integral of \( \psi \) is zero. Consequently,

\[ (u, \phi) = (u, \phi_0) \int_I \phi(x) \, dx, \quad (5.51) \]

hence \( u \) is equal to the constant \( (u, \phi_0) \).

We next consider the case where \( \Omega \) is a product of intervals: \( \Omega = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_m, b_m) \). In this case, let \( \phi_i \in \mathcal{D}(a_i, b_i) \) be a one-dimensional test function with unit integral. An arbitrary \( \phi \in \mathcal{D}(\Omega) \) is now decomposed as follows:

\[ \phi(x_1, \ldots, x_m) = \phi_1(x_1) \int_{a_1}^{b_1} \phi(s_1, x_2, \ldots, x_m) \, ds_1 + \psi_1(x_1, \ldots, x_m). \quad (5.52) \]

The function \( \psi_1 \) now has the property that

\[ \int_{a_1}^{b_1} \psi_1(x_1, x_2, \ldots, x_m) \, dx_1 = 0 \quad (5.53) \]

for every \( (x_2, \ldots, x_m) \); hence

\[ \int_{a_1}^{x_1} \psi_1(s, x_2, \ldots, x_m) \, ds \quad (5.54) \]
is again a test function. Since \( \partial u / \partial x_1 = 0 \), it follows that \((u, \psi_1) = 0 \). Next, we write
\[
\phi_1(x_1) \int_{a_1}^{b_1} \phi(s_1, x_2, \ldots, x_m) \, ds_1
= \phi_1(x_1) \phi_2(x_2) \int_{a_1}^{b_1} \int_{a_2}^{b_2} \phi(s_1, s_2, x_3, \ldots, x_m) \, ds_2 \, ds_1
+ \phi_1(x_1) \psi_2(x_2, \ldots, x_m),
\]
where now
\[
\int_{a_2}^{b_2} \psi_2(x_2, \ldots, x_m) \, dx_2 = 0,
\]
and hence \((u, \phi_1 \psi_2) = 0 \). Proceeding thusly, we finally obtain
\[
(u, \phi) = (u, \phi_1 \phi_2 \cdots \phi_m) \int_{\Omega} \phi(x) \, dx,
\]
i.e., \(u\) is a constant.

For general \(\Omega\), it follows from the result just proved that every point has a neighborhood in which \(u\) is constant, and of course the constants must be the same if two neighborhoods overlap (Problem 5.4). The rest follows from Problem 5.7.

We next consider the existence of a primitive. Of course, we cannot define a definite integral of a generalized function. Nevertheless, primitives can be shown to exist.

**Theorem 5.50.** Let \(I = (a, b)\) be an open interval in \(\mathbb{R}\) and let \(f \in \mathcal{D}'(I)\). Then there exists \(u \in \mathcal{D}'(I)\) such that \(u' = f\). The primitive \(u\) is unique up to a constant.

**Proof.** The uniqueness part is clear from the previous theorem. To construct a primitive, we use the decomposition (5.50)
\[
\phi(x) = \phi_1(x) \int_I \phi(s) \, ds + \psi(x),
\]
and we let
\[
\chi(x) = \int_a^x \psi(y) \, dy.
\]
We then define
\[
(u, \phi) = C \int_I \phi(x) \, dx - (f, \chi),
\]
where \(C\) is an arbitrary constant. If \(\phi = \eta'\), then \(\int_I \phi(x) \, dx = 0\) and \(\psi = \phi\); hence \(\chi = \eta\). We thus find
\[
(u, \eta') = -(f, \eta);
\]
hence \( u' = f \).

The multidimensional result that any curl-free vectorfield on a simply connected domain is a gradient can also be extended to distributions; the proof is considerably more complicated than in the one-dimensional case and will not be given here.

The most elementary technique of solving an ODE is based on reducing it to the form \( y' = f \); this is why solving an ODE is referred to as "integrating" it. Such procedures also work for distributional solutions. Consider, for example, the ODE

\[
y' = a(x)y + f(x). \tag{5.62}
\]

We assume that \( a \in C^\infty(\mathbb{R}) \) and \( f \in \mathcal{D}'(\mathbb{R}) \). We can now set

\[
y(x) = z(x) \exp\left(\int_0^x a(s) \, ds\right); \tag{5.63}
\]

note that multiplication of distributions by \( C^\infty \) functions is well defined and the product rule of differentiation is easily shown to hold. We thus obtain the new ODE

\[
z'(x) = f(x) \exp\left(-\int_0^x a(s) \, ds\right). \tag{5.64}
\]

From Theorem 5.50, we know that this ODE has a one-parameter family of solutions.

In particular, if \( f \) is a continuous function, then all distributional solutions of (5.62) are the classical ones. This is not necessarily true for singular ODEs; for example both the constant 1 and the Heaviside function are solutions of \( xy' = 0 \).

Problems

5.20. Let \( f \) be a piecewise continuous function with a piecewise continuous derivative. Describe the distributional derivative of \( f \).

5.21. Find the distributional derivative of the function \( \ln |x| \).

5.22. Let \( u(x, t) = f(x + t) \), where \( f \) is any locally Riemann integrable function on \( \mathbb{R} \). Show that \( u_{tt} = u_{xx} \) in the sense of distributions.

5.23. Evaluate \( \Delta(1/r^2) \) in \( \mathbb{R}^3 \).

5.24. Show that \( e^x \cos e^x \in \mathcal{S}'(\mathbb{R}) \).

5.25. Show that \( \sum_{n \in \mathbb{N}} a_n \cos nx \) converges in the sense of distributions, provided \( |a_n| \) grows at most polynomially as \( n \to \infty \).


5.27. Discuss how the substitution (5.64) is generalized to systems of ODEs.
5.28. Show that the general solution of \( xy' = 0 \) is \( c_1 + c_2 H(x) \). Hint: Show first that if \( \phi \in D(\mathbb{R}) \) vanishes at the origin, then \( \phi(x)/x \) is a test function.

5.29. Let \( f \in \mathcal{D}'(\mathbb{R}) \) be such that \( f(x + h) = f(x) \) for every positive \( h \). Show that \( f \) is constant.

5.30. Let \( f_n \) be a convergent sequence in \( \mathcal{D}'(\mathbb{R}) \) and assume that \( F_n' = f_n \). Assume, in addition, that there is a test function \( \phi_0 \) with a nonzero integral such that the sequence \( (F_n, \phi_0) \) is bounded. Show that \( F_n \) has a convergent subsequence.

5.31. Show that an even distribution on \( \mathbb{R} \) has an odd primitive.

5.32. Assume that the support of the distribution \( f \) is the set \( \{0\} \). Show that \( f \) is a linear combination of derivatives of the delta function. Hint: Let \( n \) be as given by Lemma 5.16 and assume that \( D^\alpha \phi(0) \) vanishes for \( |\alpha| \leq n \). Let \( e \) be a test function which equals 1 for \( |x| \leq 1 \) and 0 for \( |x| \geq 2 \). Now consider the sequence \( \phi_k(x) = \phi(x)e(kx) \). Show that \( (f, \phi_k) \to 0 \) and hence \( (f, \phi) = 0 \).

5.3 Convolutions and Fundamental Solutions

The classical definition of the convolution of two functions defined on \( \mathbb{R}^m \) is

\[
f * g(x) = \int_{\mathbb{R}^m} f(x - y)g(y) \, dy.
\] (5.65)

In this section, we shall consider a generalization of the definition to generalized functions and we shall give applications to the solution of partial differential equations with constant coefficients.

5.3.1 The Direct Product of Distributions

In general, one cannot define the product of two generalized functions \( f(x) \) and \( g(x) \). However, it is always possible to multiply two generalized functions depending on different variables. That is, if \( f \in \mathcal{D}'(\mathbb{R}^p) \) and \( g \in \mathcal{D}'(\mathbb{R}^q) \), then \( f(x)g(y) \) can be defined as a distribution on \( \mathbb{R}^{p+q} \).

**Definition 5.51.** Let \( f \in \mathcal{D}'(\mathbb{R}^p), g \in \mathcal{D}'(\mathbb{R}^q) \). Then the **direct product** \( f(x)g(y) \) is the distribution on \( \mathbb{R}^{p+q} \) given by

\[
(f(x)g(y), \phi(x,y)) = (f(x), (g(y), \phi(x,y)))).
\] (5.66)

That is, we first regard \( \phi(x,y) \) as a function only of \( y \), which depends on \( x \) as a parameter. To this function we apply the functional \( g \). The result is then a real-valued function \( \psi(x) \), which obviously has compact support. It is easy to show that \( \psi \) is of class \( C^\infty \) (Problem 5.33). Hence
ψ is a test function and \((f, \psi)\) is well defined. To justify the definition, it remains to be shown that \((f(x), (g(y), \phi_n(x,y)))\) converges to zero if \(\phi_n\) converges to zero in \(\mathcal{D}(\mathbb{R}^{p+q})\). Since \(f\) is a distribution, it suffices to show that \(\psi_n := (g(y), \phi_n(x,y))\) converges to zero in \(\mathcal{D}(\mathbb{R}^p)\). If \(S_p \times S_q\) is a compact set containing the supports of all the \(\phi_n\), then \(S_p\) contains the supports of all the \(\psi_n\). It remains to be shown that \(\psi_n\) and all its derivatives converge uniformly to zero. Let \(\alpha\) be a \(p\)-dimensional multi-index and let \(\beta = (\alpha, 0, \ldots, 0)\). Assume that \(D^\alpha \psi_n\) does not converge uniformly to zero. After choosing a subsequence, we may assume that there is a sequence of points \(x_n\) such that

\[
|D^\alpha \psi_n(x_n)| = |(g(y), D^\beta \phi_n(x_n,y))| \geq \epsilon. \tag{5.67}
\]

But since the \(\phi_n\) converge to zero with all their derivatives, the same is true for the sequence \(\chi_n(y) := D^\beta \phi_n(x_n,y)\). Hence \(\chi_n\) converges to zero in \(\mathcal{D}(\mathbb{R}^q)\) and therefore \((g, \chi_n)\) converges to zero, a contradiction.

**Example 5.52.** As a simple example of a direct product, we note that

\[
\delta(x)\delta(y) = \delta(x,y).
\]

If \(\phi(x,y)\) has the special form \(\phi_1(x)\phi_2(y)\), we obtain

\[
(f(x)g(y), \phi(x,y)) = (f, \phi_1)(g, \phi_2). \tag{5.68}
\]

Linear combinations of the form \(\phi_1(x)\phi_2(y)\) are actually dense in \(\mathcal{D}(\mathbb{R}^{p+q})\). To see this, let \(\phi\) have support in the set \(Q := \{|x| \leq a, |y| \leq a\}\). By the Weierstraß approximation theorem (see Section 2.3.3), there is a sequence of polynomials which converges to \(\phi\) uniformly on the set \(Q' := \{|x| \leq 2a, |y| \leq 2a\}\). Moreover, the argument used in the proof of the theorem also shows that the derivatives of the polynomials converge uniformly to those of \(\phi\) on \(Q'\). We can thus choose polynomials \(p_n\) in such a way that on \(Q'\) we have

\[
|D^\alpha p_n - D^\alpha \phi| \leq \frac{1}{n}, \quad \forall \, |\alpha| \leq n. \tag{5.69}
\]

Let now \(b_1(x), b_2(y)\) be fixed test functions which are equal to 1 for \(|x| \leq a\) \((|y| \leq a)\) and equal to 0 for \(|x| \geq 2a, |y| \geq 2a\). Then the sequence

\[
\phi_n(x,y) := b_1(x)b_2(y)p_n(x,y) \tag{5.70}
\]

converges to \(\phi\) in \(\mathcal{D}(\mathbb{R}^{p+q})\).

This fact and continuity can be used to show properties of the direct product by verifying them only on the restricted set of test functions of the form \(\phi_1(x)\phi_2(y)\). One immediate conclusion is that in the definition we can evaluate \(f\) and \(g\) in the opposite order, i.e., we also have

\[
(f(x)g(y), \phi(x,y)) = (g(y), (f(x), \phi(x,y)));
\]

we express this fact by the suggestive notation

\[
f(x)g(y) = g(y)f(x). \tag{5.72}
\]
Another obvious property is the associative law
\[ f(x)(g(y)h(z)) = (f(x)g(y))h(z). \] (5.73)

5.3.2 Convolution of Distributions
Let \( f \) and \( g \) be continuous functions on \( \mathbb{R}^m \) which decay rapidly at infinity. We then have the following identity:
\[
(f * g, \phi) = \int_{\mathbb{R}^m} (f * g)(x)\phi(x) \, dx
= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(x-y)g(y)\phi(x) \, dx \, dy
= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} f(x)g(y)\phi(x+y) \, dx \, dy.
\] (5.74)
This identity is used as the definition of the convolution of two distributions.

"Definition" 5.53. Let \( f, g \in \mathcal{D}'(\mathbb{R}^m) \). Then the convolution of \( f \) and \( g \) is defined by
\[
(f * g, \phi) = (f(x)g(y), \phi(x + y)).
\] (5.75)
The quotes are meant to draw attention to the fact that this does not make any sense. We defined the direct product \( f(x)g(y) \) as an element of \( \mathcal{D}'(\mathbb{R}^{2m}) \), but \( \phi(x+y) \) is not in \( \mathcal{D}(\mathbb{R}^{2m}) \); it does not have compact support. Indeed, the convolution of arbitrary distributions cannot be defined in a rational manner. There are, however, special cases where a meaning can be given to (5.75). In the simplest such case, the support of \( \phi(x+y) \) has a compact intersection with the support of \( f(x)g(y) \). If this is the case, we may replace \( \phi(x+y) \) by any test function which agrees with \( \phi(x+y) \) in a neighborhood of \( \text{supp}(f(x)g(y)) \). In particular, (5.75) is meaningful under either of the following conditions:

1. Either \( f \) or \( g \) has compact support.
2. In one dimension, the supports of \( f \) and \( g \) are bounded from the same side (e.g., \( f = 0 \) for \( x < a \) and \( g = 0 \) for \( x < b \)).

From the corresponding properties of the direct product, it follows that convolution is commutative and associative where it is defined.

Let us consider some special cases:

1. We have
\[
(\delta * f, \phi) = (\delta(x)f(y), \phi(x+y))
= (f(y), (\delta(x), \phi(x+y)))
= (f(y), \phi(y))
= (f, \phi),
\] (5.76)
i.e., $\delta * f = f$.

2. Let us consider $f * \psi$, where $\psi \in \mathcal{D}(\mathbb{R}^m)$. We have

$$
(f * \psi, \phi) = (f(x)\psi(y), \phi(x + y))
$$

$$
= \left( f(x), \int_{\mathbb{R}^m} \psi(y) \phi(x + y) \, dy \right)
$$

$$
= \left( f(x), \int_{\mathbb{R}^m} \psi(y - x) \phi(y) \, dy \right)
$$

$$
= \int_{\mathbb{R}^m} (f(x), \psi(y - x)) \phi(y) \, dy.
$$

(5.77)

In the last step, we have used the continuity of the functional $f$ to take it under the integral; see Problem 5.36. Hence $f * \psi(y)$ is equal to the function $(f(x), \psi(y - x))$. This function is of class $C^\infty$, and if $f$ has compact support, it is a test function.

We next consider differentiation of a convolution. By definition, we have

$$
(D^\alpha(f * g), \phi) = (-1)^{|\alpha|} (f * g, D^\alpha \phi)
$$

$$
= (-1)^{|\alpha|} (g(y), (f(x), D^\alpha \phi(x + y)))
$$

$$
= (g(y), (D^\alpha f(x), \phi(x + y)))
$$

$$
= (D^\alpha f * g, \phi).
$$

(5.78)

Thus $D^\alpha(f * g) = D^\alpha f * g$, and using commutativity, we also find $D^\alpha(f * g) = f * D^\alpha g$. A convolution is differentiated by differentiating either one of the factors.

The following lemma expresses continuity of the operation of convolution.

**Lemma 5.54.** Assume that $f_n \to f$ in $\mathcal{D}'(\mathbb{R}^m)$ and that one of the following holds:

1. The supports of all the $f_n$ are contained in a common compact set;

2. $g$ has compact support;

3. $m = 1$ and the supports of the $f_n$ and of $g$ are bounded on the same side, independently of $n$.

Then $f_n * g \to f * g$ in $\mathcal{D}'(\mathbb{R}^m)$.

The proof is left as an exercise (cf. Problem 5.37). A consequence is the following theorem.

**Theorem 5.55.** $\mathcal{D}(\mathbb{R}^m)$ is dense in $\mathcal{D}'(\mathbb{R}^m)$.

**Proof.** We first show that distributions of compact support are dense. To see this, simply let $e_n$ be a test function which equals 1 on the set $\{||x|| \leq n\}$. Then $e_n f \to f$ for every $f \in \mathcal{D}'(\mathbb{R}^m)$, and the support of $e_n f$ is contained in the support of $e_n$, hence compact.
It therefore suffices to show that distributions of compact support are limits of test functions. Let \( f \) be a distribution of compact support, and let \( \phi_n \) be a delta-convergent sequence of test functions; we may for example choose the sequence \( \phi_n = C(1/n)^{-1} \phi_{0,1/n} \), where \( \phi_{0,\epsilon} \) and \( C(\epsilon) \) are defined by (5.6) and (5.8). Then \( \phi_n * f \) is a test function, and by the previous lemma \( \phi_n * f \) converges to \( \delta * f = f \).

5.3.3 Fundamental Solutions

**Definition 5.56.** Let \( L(D) \) be a differential operator with constant coefficients. Then a fundamental solution for \( L \) is a distribution \( G \in D'(\mathbb{R}^m) \) satisfying the equation \( L(D)G = \delta \).

Of course, fundamental solutions are unique only up to a solution of the homogeneous equation \( L(D)u = 0 \); in choosing a specific fundamental solution one often selects the one with the “nicest” behavior at infinity. The significance of the fundamental solution lies in the fact that

\[
L(D)(G * f) = (L(D)G) * f = \delta * f = f, \quad (5.79)
\]

provided that the convolution \( G * f \) is defined. If, for example, \( f \) has compact support, then \( G * f \) is a solution of the equation \( L(D)u = f \).

The construction of fundamental solutions for general operators with constant coefficients is quite complicated, and we shall limit our discussion to some important examples.

**Example 5.57.** Ordinary differential equations. We seek a solution to the ODE

\[
a_n G^{(n)}(x) + \cdots + a_0 G(x) = \delta(x). \quad (5.80)
\]

For both positive and negative \( x \), \( G \) must agree with a solution of the homogeneous ODE. That is, if \( u_1(x), \ldots, u_n(x) \) are a complete set of solutions for the homogeneous ODE, then we must have

\[
G(x) = \begin{cases} 
\alpha_1 u_1(x) + \cdots + \alpha_n u_n(x) & x > 0 \\
\beta_1 u_1(x) + \cdots + \beta_n u_n(x) & x < 0.
\end{cases} \quad (5.81)
\]

We can now satisfy (5.80) by requiring that all derivatives of \( G \) up to the \((n - 2)\)nd are continuous at 0, but the \((n - 1)\)st derivative has a jump of magnitude \( 1/a_n \). With \( \gamma_i = \alpha_i - \beta_i \), this yields the system

\[
\gamma_1 u_1(0) + \cdots + \gamma_n u_n(0) = 0, \\
\gamma_1 u_1'(0) + \cdots + \gamma_n u_n'(0) = 0, \\
\vdots \\
\gamma_1 u_1^{(n-2)}(0) + \cdots + \gamma_n u_n^{(n-2)}(0) = 0, \\
\gamma_1 u_1^{(n-1)}(0) + \cdots + \gamma_n u_n^{(n-1)}(0) = \frac{1}{a_n}.
\]

\[\Box\]
The determinant of this system is the Wronskian of the complete set of solutions $u_i$ and is hence nonzero.

**Example 5.58. Laplace’s equation.** We now seek a solution of the equation

\[ \Delta G(x) = \delta(x) \quad (5.83) \]

on $\mathbb{R}^m$. Of course this makes $G$ a solution of the homogeneous Laplace equation except at the origin. Moreover, since $\delta$ is radially symmetric, it is natural to seek a radially symmetric $G$. For radially symmetric functions, Laplace’s equation reduces to

\[ G''(r) + \frac{m-1}{r} G'(r) = 0, \quad r > 0, \quad (5.84) \]

and we obtain the solution

\[ G(r) = \begin{cases} c_1 + c_2 r^{2-m} & m \geq 3 \\ c_1 + c_2 \ln r & m = 2. \end{cases} \quad (5.85) \]

We can now satisfy (5.83) by an appropriate choice of $c_2$. For $m = 3$, we did this calculation in Example 5.46 of the previous section; the result was $c_2 = -1/4\pi$. The general result is obtained in an analogous fashion; one finds the fundamental solutions

\[ G(x) = \begin{cases} -r^{2-m}/(m-2)\Omega_m & m \geq 3 \\ \ln r/2\pi & m = 2. \end{cases} \quad (5.86) \]

Here $\Omega_m = 2\pi^{m/2}/\Gamma(m/2)$ is the surface area of the unit sphere (cf. Problem 4.16).

**Example 5.59. The heat equation.** For equations which are naturally posed as initial-value problems, a different definition of fundamental solution is used. Consider the equation

\[ u_t(x, t) = L(D)u(x, t), \quad x \in \mathbb{R}^m, \quad t > 0, \quad (5.87) \]

where $L$ is a constant coefficient differential operator on $\mathbb{R}^m$. Instead of regarding $u$ as a distribution on $\mathbb{R}^m \times (0, \infty)$, we shall in the following regard $u$ as a distribution on $\mathbb{R}^m$, depending on $t$ as a parameter. We say that $u$ depends continuously on $t$ if

\[ \int_{\mathbb{R}^m} u(x, t)\phi(x) \, dx \quad (5.88) \]

is continuous in $t$ for every $\phi \in \mathcal{D}(\mathbb{R}^m)$ and we say that $u$ is differentiable with respect to $t$ if (5.88) is differentiable for every $\phi$. If $u$ is differentiable with respect to $t$, the derivative is also a distribution on $\mathbb{R}^m$; this follows from the representation of the derivative as a limit of difference quotients and the completeness of the space of distributions.
If $u(x, t)$ is a distribution on $\mathbb{R}^m$ depending continuously on $t > 0$, we can also regard $u$ as a distribution on $\mathbb{R}^m \times (0, \infty)$. This is because every test function $\phi(x, t) \in \mathcal{D}(\mathbb{R}^m \times (0, \infty))$ can also be thought of as a test function on $\mathbb{R}^m$ which depends continuously on the parameter $t$. Because of Lemma 5.32, this makes
\[
\int_{\mathbb{R}^m} u(x, t) \phi(x, t) \, dx \tag{5.89}
\]
a continuous function of $t$; hence
\[
\int_0^\infty \int_{\mathbb{R}^m} u(x, t) \phi(x, t) \, dx \, dt \tag{5.90}
\]
exists. Hence $u$ defines a linear functional on $\mathcal{D}(\mathbb{R}^m \times (0, \infty))$; the continuity of this functional can be deduced, for example, by representing the outer integral in (5.90) as a limit of Riemann sums and using the completeness of the space of distributions.

We are now ready to define a fundamental solution.

**Definition 5.60.** We call $G : [0, \infty) \to \mathcal{D}'(\mathbb{R}^m)$ a fundamental solution of (5.87) if $G$ is continuously differentiable on $[0, \infty)$, and moreover, $G$ satisfies (5.87) with the initial condition $G(x, 0) = \delta(x)$.

In this definition, we think of $u_t$ in (5.87) as differentiation with respect to the parameter $t$. Nevertheless, it is easy to show that $G$ also satisfies (5.87) in the sense of distributions on $\mathbb{R}^m \times (0, \infty)$.

A solution of the inhomogeneous problem
\[
u_t = L(D)u + f(x, t), \quad u(x, 0) = u_0(x), \tag{5.91}\]
where $f$ and $u_0$ have compact support, and $f$ is continuous from $[0, \infty)$ to $\mathcal{D}'(\mathbb{R}^m)$, can now be represented as follows:
\[
u(x, t) = \int_{\mathbb{R}^m} G(x - y, t)u_0(y) \, dy + \int_0^t \int_{\mathbb{R}^m} G(x - y, t - s)f(y, s) \, dy \, ds. \tag{5.92}\]
The reader should verify that this is indeed a solution (cf. Problem 5.41).

We now consider the heat equation in one space dimension. Problem 1.24 asks for the solution of the problem
\[
u_t = u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \quad u(x, 0) = H(x), \quad x \in \mathbb{R}. \tag{5.93}\]
The solution can be found by the ansatz $u(x, t) = \phi(x/\sqrt{t})$, which reduces the problem to an ODE. The result is
\[
u(x, t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{t}} \exp\left(-\frac{v^2}{4}\right) \, dv. \tag{5.94}\]
To obtain the fundamental solution, we simply need to differentiate with respect to \( x \). We thus obtain
\[
G(x, t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right). \tag{5.95}
\]

The fundamental solution for the heat equation in several dimensions can be obtained as a direct product:
\[
G(x, t) = \left(\frac{1}{2\sqrt{\pi t}}\right)^m \exp\left(-\frac{|x|^2}{4t}\right). \tag{5.96}
\]

**Example 5.61. The wave equation.** For second-order equations
\[
u_{tt} = L(D)u \tag{5.97}
\]
we define the fundamental solution \( G \) as a twice continuously differentiable function \([0, \infty) \to \mathcal{D}'(\mathbb{R}^m)\) such that \( G \) satisfies (5.97) with the initial conditions \( G(x, 0) = 0, \ G_t(x, 0) = \delta(x) \). The solution of the inhomogeneous problem
\[
u_{tt} = L(D)u + f(x, t), \ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x) \tag{5.98}
\]
is then represented by
\[
u(x, t) = \int_{\mathbb{R}^m} G_t(x - y, t)u_0(y) \, dy + \int_{\mathbb{R}^m} G(x - y, t)u_1(y) \, dy
+ \int_0^t \int_{\mathbb{R}^m} G(x - y, t - s)f(y, s) \, dy \, ds. \tag{5.99}
\]
For the wave equation in one space dimension,
\[
G(x, t) = \begin{cases} 
1/2 & |x| < t \\
0 & |x| \geq t 
\end{cases} \tag{5.100}
\]
is a fundamental solution. Indeed, it is obvious that \( G(x, 0) = 0 \) and from the representation \( G(x, t) = (H(x+t) - H(x-t))/2 \) it follows that \( G \) satisfies the wave equation and that \( G_t(x, t) = (\delta(x+t) + \delta(x-t))/2 \), which equals \( \delta(x) \) for \( t = 0 \). The fundamental solution for the wave equation in several dimensions will be discussed in the next section. We draw attention to the fact that the fundamental solutions for the Laplace and heat equations are (apart from the singularity at the origin) \( C^\infty \) functions, but that of the wave equation is not. This has important implications for the regularity of solutions.

**Problems**

**5.33.** Let \( \phi(x, y) \in \mathcal{D}(\mathbb{R}^{p+q}), \ g \in \mathcal{D}'(\mathbb{R}^q) \). Show that
\[
\psi(x) := (g(y), \phi(x, y))
\]
is in \( \mathcal{D}(\mathbb{R}^p) \).
5.34. Show that the direct product can also be defined in $S'$.

5.35. Let $F$ and $G$ be the supports of $f$ and $g$. Show that the support of the direct product is $F \times G$.

5.36. Let $\phi, \psi \in \mathcal{D}(\mathbb{R}^m)$. Show that $\int_{\mathbb{R}^m} \psi(y-x)\phi(y) \, dy$ defines an element of $\mathcal{D}(\mathbb{R}^m)$. Moreover, show that, in the sense of convergence in $\mathcal{D}(\mathbb{R}^m)$, the integral is a limit of Riemann sums.

5.37. Prove Lemma 5.54.

5.38. Discuss how the proof of Theorem 5.55 needs to be modified to show that $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}'(\Omega)$. Also show that $\mathcal{D}(\mathbb{R}^m)$ is dense in $S'(\mathbb{R}^m)$.

5.39. Show that the direct product is jointly continuous in its factors.

5.40. Find a fundamental solution for the biharmonic equation $\Delta \Delta G = \delta(x)$ on $\mathbb{R}^m$.

5.41. Prove that (5.92) is indeed a solution of (5.91).

5.42. Let $G$ be the fundamental solution corresponding to the initial-value problem of $u_t = L(D)u$. Show that the functional

$$F : \phi \rightarrow \int_0^\infty \int_{\mathbb{R}^m} G(x,t)\phi(x,t) \, dx \, dt$$

defines a distribution on $\mathbb{R}^{m+1}$ and that this distribution satisfies the equation $F_t - L(D)F = \delta(x,t)$.

5.43. Specialize (5.99) to the one-dimensional wave equation.

5.44. Let $f$ be a distribution with compact support and let $P$ be a polynomial. Show that $P * f$ is a polynomial.

5.4 The Fourier Transform

5.4.1 Fourier Transforms of Test Functions

**Definition 5.62.** The Fourier transform of a continuous, absolutely integrable function $f : \mathbb{R}^m \to \mathbb{C}$ is defined by

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\xi \cdot x} f(x) \, dx. \quad (5.102)$$

In particular, this defines the Fourier transform of every $f \in S(\mathbb{R}^m)$. In fact, we have the following result.

---

4Definitions in the literature differ as to whether or not the minus sign is included in the exponent and whether the factor $(2\pi)^{-m/2}$ is included.
**Theorem 5.63.** If \( f \in \mathcal{S}(\mathbb{R}^m) \), then \( \hat{f} \in \mathcal{S}(\mathbb{R}^m) \). Moreover, the mapping \( \mathcal{F} \) is continuous from \( \mathcal{S}(\mathbb{R}^m) \) into itself.

**Proof.** If \( f \in \mathcal{S}(\mathbb{R}^m) \), then clearly \( \hat{f} \) is a continuous, bounded function; moreover, if \( f_n \to 0 \) in \( \mathcal{S}(\mathbb{R}^m) \), then \( \hat{f}_n \to 0 \) uniformly. The rest follows from the identities

\[
D^\alpha \hat{f}(\xi) = \mathcal{F}[(-ix)^\alpha f](\xi), \tag{5.103}
\]

\[
(i\xi)^\alpha \hat{f}(\xi) = \mathcal{F}[D^\alpha f](\xi). \tag{5.104}
\]

The first of these identities is obtained by differentiating under the integral, the second by integration by parts. \( \square \)

Equation (5.104) is the principal reason why Fourier transforms are important; if \( L(D) \) is a differential operator with constant coefficients, then

\[
\mathcal{F}[L(D)u] = L(i\xi)\mathcal{F}[u], \tag{5.105}
\]

where \( L(i\xi) \) is the symbol of \( L \) defined in Section 2.1. Partial differential equations with constant coefficients can therefore be transformed into algebraic equations by Fourier transform. Of course, knowing the Fourier transform of a solution is of little use, unless we know how to invert the transform. This is addressed by the next theorem.

**Theorem 5.64.** Let \( g \in \mathcal{S}(\mathbb{R}^m) \). Then there is a unique \( f \in \mathcal{S}(\mathbb{R}^m) \) such that \( g = \mathcal{F}[f] \). Furthermore, the inverse Fourier transform of \( g \) is given by the formula

\[
f(x) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{i\xi \cdot x} g(\xi) \, d\xi. \tag{5.106}
\]

Except for the minus sign in the exponent, the formula for the inverse Fourier transform agrees with that for the Fourier transform itself. To evaluate the integrals arising in Fourier transforms, complex contour deformations are often useful; for examples, see Problem 5.46.

**Proof.** Let \( Q_M = [-M, M]^m \), and let \( f \) be given by (5.106). Then we find

\[
\hat{f}(\xi) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\xi \cdot x} f(x) \, dx
\]

\[
= (2\pi)^{-m} \int_{\mathbb{R}^m} e^{-i\xi \cdot x} \int_{\mathbb{R}^m} e^{i\eta \cdot x} g(\eta) \, d\eta \, dx
\]

\[
= (2\pi)^{-m} \lim_{M \to \infty} \int_{Q_M} \int_{\mathbb{R}^m} e^{i(\eta - \xi) \cdot x} g(\eta) \, d\eta \, dx
\]

\[
= (2\pi)^{-m} \lim_{M \to \infty} \int_{\mathbb{R}^m} \int_{Q_M} e^{i(\eta - \xi) \cdot x} g(\eta) \, dx \, d\eta
\]

\[
= \pi^{-m} \lim_{M \to \infty} \int_{\mathbb{R}^m} \prod_{i=1}^{m} \frac{\sin M(\eta_i - \xi_i)}{\eta_i - \xi_i} g(\eta) \, d\eta.
\]
As we have seen in Example 5.48, the limit of $\sin M(\eta_i - \xi_i)/(\eta_i - \xi_i)$ as $M \to \infty$ is $\pi \delta(\eta_i - \xi_i)$ in the sense of distributions, and also in the sense of tempered distributions. Using this fact and the continuity of the direct product, we find $\hat{f}(\xi) = g(\xi)$, i.e., the Fourier transform of $f$ is indeed $g$.

An analogous calculation shows that if $g = \hat{h}$ for some function $h \in \mathcal{S}(\mathbb{R}^m)$, then $h = f$ as given by (5.106).

An important property of the Fourier transform is that it preserves the inner product.

**Theorem 5.65.** Let $f, \phi \in \mathcal{S}(\mathbb{R}^m)$. Then $(\hat{f}, \hat{\phi}) = (f, \phi)$.

**Proof.** We have

$$
(f, \phi) = \int_{\mathbb{R}^m} \overline{f(x)}\phi(x) \, dx
= (2\pi)^{-m/2} \int_{\mathbb{R}^m} \overline{f(x)} \int_{\mathbb{R}^m} \hat{\phi}(\xi) e^{i\xi \cdot x} \, d\xi \, dx
= (2\pi)^{-m/2} \int_{\mathbb{R}^m} \hat{\phi}(\xi) \int_{\mathbb{R}^m} \overline{f(x)} e^{-i\xi \cdot x} \, dx \, d\xi
= \int_{\mathbb{R}^m} \overline{\hat{f}(\xi)} \hat{\phi}(\xi) \, d\xi = (\hat{f}, \hat{\phi}).
$$

(5.108)

This completes the proof. \(\square\)

### 5.4.2 Fourier Transforms of Tempered Distributions

The previous theorem motivates the definition of the Fourier transform of a tempered distribution.

**Definition 5.66.** Let $f \in \mathcal{S}'(\mathbb{R}^m)$. Then the **Fourier transform** of $f$ is defined by the functional

$$
(\mathcal{F}[f], \phi) = (f, \mathcal{F}^{-1}[\phi]), \quad \phi \in \mathcal{S}(\mathbb{R}^m).
$$

(5.109)

It is clear from the definition that $\mathcal{F}$ is a continuous mapping from $\mathcal{S}'(\mathbb{R}^m)$ into itself. It is also easy to check that the formulas (5.103) and (5.104) still hold in $\mathcal{S}'(\mathbb{R}^m)$; the same is true for the inversion formula (5.106), which can be restated as

$$
\mathcal{F}\mathcal{F}[f(x)] = f(-x);
$$

(5.110)

this form has meaning for generalized functions.

We shall now consider a number of examples.

**Example 5.67.** We have

$$
(\mathcal{F}[\delta], \phi) = (\delta, \mathcal{F}^{-1}[\phi]) = \mathcal{F}^{-1}[\phi](0)
= (2\pi)^{-m/2} \int_{\mathbb{R}^m} \phi(x) \, dx,
$$

(5.111)
i.e., the Fourier transform of $\delta$ is the constant $(2\pi)^{-m/2}$.

**Example 5.68.** We have

$$
(F[1], \phi) = (1,\mathcal{F}^{-1}[\phi]) = \int_{\mathbb{R}^m} \mathcal{F}^{-1}[\phi](x) \, dx
$$

(5.112)

$$
= (2\pi)^{m/2} \mathcal{F}\mathcal{F}^{-1}[\phi](0) = (2\pi)^{m/2} \phi(0),
$$

i.e., the Fourier transform of 1 is $(2\pi)^{m/2}\delta$. From (5.103), (5.104), it is now clear that the Fourier transforms of polynomials are linear combinations of derivatives of the delta function and vice versa.

**Example 5.69.** A calculation similar to that in Example 5.68 shows that the Fourier transform of $\exp(i\eta \cdot x)$ (viewed as a function of $x$) is $(2\pi)^{m/2}\delta(\xi - \eta)$. If $f$ is a periodic distribution given by a Fourier series

$$
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},
$$

(5.113)

we find that

$$
\mathcal{F}[f](\xi) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} c_n \delta(\xi - n).
$$

(5.114)

**Example 5.70.** Let $f$ be a distribution with compact support. Then we can define $(f, \phi)$ for any $\phi \in C^\infty(\mathbb{R}^m)$; we set $(f, \phi) = (f, \phi_0)$, where $\phi_0$ is any element of $\mathcal{D}(\mathbb{R}^m)$ which agrees with $\phi$ in a neighborhood of the support of $f$. It follows from the definition of the support that this definition does not depend on the choice of $\phi_0$. In particular, this defines $f$ as an element of $\mathcal{S}'(\mathbb{R}^m)$. We claim now that $\mathcal{F}[f]$ is the function

$$
\mathcal{F}[f](\xi) = (2\pi)^{-m/2} (\bar{f}(\xi), e^{-i\xi \cdot x}).
$$

(5.115)

Here $(\bar{f}, \phi)$ is defined as the complex conjugate of $(f, \phi)$. To verify the claim, we must show that, for any $\phi \in \mathcal{S}(\mathbb{R}^m)$, we have

$$
(2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} (f(x), e^{i\xi \cdot x}) \phi(\xi) \, d\xi = (f, \mathcal{F}^{-1}[\phi])
$$

$$
= (2\pi)^{-\frac{m}{2}} \left( f, \int_{\mathbb{R}^m} e^{i\xi \cdot x} \phi(\xi) \, d\xi \right).
$$

(5.116)

That is, we have to justify taking $f$ under the integral, which is accomplished in the usual way by approximating the integral by finite sums. We note that (5.115) defines an entire function of $\xi \in \mathbb{C}^m$. The fact that a distribution of compact support has finite order (Lemma 5.16) implies that for real arguments this function has polynomial growth.

**Example 5.71.** Fourier transforms which cannot be defined classically as an integral can often be determined as limits of regularizations. As an ex-
ample, we consider the Heaviside function $H(x)$. Clearly, we cannot define
the Fourier transform as
\[
\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\xi x} \, dx.
\] (5.117)

Observe, however, that
\[
H(x) = \lim_{\epsilon \to 0^+} H(x)e^{-\epsilon x}
\] (5.118)
in the sense of tempered distributions, and consequently
\[
\mathcal{F}[H](\xi) = \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0^+} \int_0^\infty e^{-\epsilon x - i\xi x} \, dx = \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{2\pi}} \frac{1}{\epsilon + i\xi}.
\] (5.119)

We can evaluate this limit a little more explicitly by applying it to a test
function. For any $\delta > 0$, we have
\[
\lim_{\epsilon \to 0^+} \int_{|x| > \delta} \frac{\phi(\xi)}{i\xi} \, d\xi = \int_{|\xi| > \delta} \frac{\phi(\xi)}{i\xi} \, d\xi + \lim_{\epsilon \to 0^+} \int_{-\delta}^{\delta} \frac{\phi(0)}{i\xi} \, d\xi
\]
\[
= \int_{|\xi| > \delta} \frac{\phi(\xi)}{i\xi} \, d\xi + \int_{-\delta}^{\delta} \frac{\phi(\xi) - \phi(0)}{i\xi} \, d\xi + \pi \phi(0).
\] (5.120)

We conclude that
\[
\mathcal{F}[H] = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{i\xi} + \pi \delta \right),
\] (5.121)
where $1/(i\xi)$ is interpreted as a principal value.

**Example 5.72.** Let $f$ be any continuous function which has polynomial
growth at infinity. Then, in the sense of tempered distributions, $f$ is the
limit as $M \to \infty$ of
\[
f_M(x) = \begin{cases} f(x), & |x| \leq M \\ 0, & |x| > M. \end{cases}
\] (5.122)

As a consequence, we find that, in the sense of tempered distributions,
\[
\hat{f}(\xi) = (2\pi)^{-m/2} \lim_{M \to \infty} \int_{|x| \leq M} f(x)e^{-i\xi x} \, dx.
\] (5.123)

In particular, if $f$ is integrable at infinity, the Fourier transform of $f$
as a distribution agrees with the ordinary Fourier transform. Another
way to evaluate the Fourier transform of functions with polynomial
growth is therefore to approximate them by integrable functions, such as
$f(x) \exp(-\epsilon|x|^2)$. See Problem 5.48 for examples.
Example 5.73. Let \( \delta(r-a) \) represent a uniform mass distribution on the sphere of radius \( a \), i.e.,
\[
(\delta(r-a), \phi) = \int_{|x|=a} \phi(x) \, dS. \tag{5.124}
\]
(Of course, this is not consistent with our previous use of \( \delta \) as a distribution on \( \mathbb{R}^m \), but it is a standard abuse of notation with which the reader should become accustomed.) Then the Fourier transform of \( \delta(r-a) \) is given by (5.115)
\[
\mathcal{F}[\delta(r-a)](\xi) = (2\pi)^{-m/2} \int_{|x|=a} e^{-i\xi \cdot x} \, dS. \tag{5.125}
\]
We want to evaluate this expression for \( m = 3 \). We use polar coordinates with the axis aligned with the direction of \( \xi \) so that \( \xi \cdot x = a|\xi| \cos \theta \); we shall use \( \rho \) to denote \( |\xi| \). We thus find
\[
\mathcal{F}[\delta(r-a)](\xi) = (2\pi)^{-3/2} a^2 \int_0^\pi \int_0^{2\pi} e^{-ia\rho \cos \theta} \sin \theta \, d\phi \, d\theta = \sqrt{\frac{2}{\pi}} a \sin a \rho. \tag{5.126}
\]
Example 5.74. The Fourier transform of a direct product is the direct product of the Fourier transforms. To show this, it suffices to prove agreement for a dense set of test functions. We have
\[
(\hat{f}(\xi) \hat{g}(\eta), \hat{\phi}(\xi) \hat{\psi}(\eta)) = (\hat{f}, \hat{\phi})(\hat{g}, \hat{\psi}) = (f, \phi)(g, \psi) = (f(x)g(y), \phi(x)\psi(y)). \tag{5.127}
\]

5.4.3 The Fundamental Solution for the Wave Equation

The Fourier transform is obviously useful in obtaining fundamental solutions. If \( L(D) \) is a constant coefficient operator, then the equation \( L(D)u = \delta \) is transformed to \( L(i\xi)\hat{u} = (2\pi)^{-m/2} \), i.e., to a purely algebraic equation. We immediately obtain
\[
\hat{u}(\xi) = \frac{1}{(2\pi)^{m/2}L(i\xi)}; \tag{5.128}
\]
the only problem is that \( L(i\xi) \) may have zeros. If (5.128) has nonintegrable singularities, we have to consider appropriate regularizations. Finally, one has to compute the inverse Fourier transform of \( \hat{u}(\xi) \); this step is not necessarily easy.

Similarly, the Fourier transform can be used to find fundamental solutions for initial-value problems; we shall now do so for the wave equation in \( \mathbb{R}^3 \). The problem
\[
G_{tt} = \Delta G, \quad G(x,0) = 0, \quad G_t(x,0) = \delta(x). \tag{5.129}
\]
is Fourier transformed in the spatial variables only; i.e., we define
\[ \hat{G}(\xi, t) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\xi \cdot x} G(x, t) \, dx, \]
and apply the same type of transform to (5.129). The result is an ODE in the variable \( t \),
\[ \hat{G}_{tt}(\xi, t) = -|\xi|^2 \hat{G}(\xi, t), \quad \hat{G}(\xi, 0) = 0, \quad \hat{G}_t(\xi, 0) = (2\pi)^{-3/2}. \] (5.131)
With \( |\xi| = \rho \), the solution is easily obtained as
\[ \hat{G}(\xi, t) = (2\pi)^{-3/2} \frac{\sin \rho t}{\rho}. \] (5.132)
Using Example 5.73 above, we find
\[ G(x, t) = \frac{\delta(r - t)}{4\pi t}. \] (5.133)
It can be shown that, in any odd space dimension greater than 1, the fundamental solution of the wave equation can be expressed in terms of derivatives of \( \delta(r - t) \); since there is little applied interest in solving the wave equation in more than three dimensions, we shall not prove this here. It is, however, of interest to solve the wave equation in two dimensions. In even space dimensions, it is not easy to evaluate the inverse Fourier transform of \( \sin \rho t/\rho \) directly; instead, one uses a trick known as the method of descent. This trick is based on the simple observation that any solution of the wave equation in two dimensions can be regarded as a solution in three dimensions, simply by taking the direct product with the constant function 1. The fundamental solution in two dimensions can therefore be obtained by convolution of (5.133) with \( \delta(x)\delta(y)1(z) \). Using the definition of convolution (5.75), we compute
\[ \left( \delta(x)\delta(y)1(z) \frac{\delta(r' - t)}{4\pi t}, \phi(x + x') \right) = \frac{1}{4\pi t} \int_{-\infty}^{\infty} \int_{r' = t}^{\infty} \phi(x', y', z' + z) \, dS' \, dz. \] (5.134)
With \( \psi(x, y) \) denoting \( \int_{-\infty}^{\infty} \phi(x, y, z) \, dz \), (5.134) simplifies to
\[ \frac{1}{4\pi t} \int_{r' = t}^{\infty} \psi(x', y') \, dS', \] (5.135)
and evaluation of this integral yields
\[ \frac{1}{2\pi} \int_{x^2 + y^2 \leq t^2} \frac{\psi(x, y)}{\sqrt{t^2 - x^2 - y^2}} \, dx \, dy. \] (5.136)
We have thus obtained the following fundamental solution in two space dimensions:
\[ G(x, t) = \begin{cases} \frac{1}{2\pi} \sqrt{t^2 - r^2}, & r < t \\ 0, & r \geq t. \end{cases} \] (5.137)
We note that the qualitative nature of the fundamental solution for the
heat equation does not really change with the space dimension, but the
fundamental solution of the wave equation changes dramatically. In any
number of dimensions, the support of the fundamental solution for the
wave equation is contained in $|x| \leq t$, but otherwise the fundamental solutions
look quite different. Whereas the fundamental solution in three dimensions
is supported only on the sphere $|x| = t$, the support of (5.137) fills out
the full circle. Television sets in Abbott’s Flatland [Ab] would have to be
designed quite differently from ours; in this context, see also [Mo].

5.4.4 Fourier Transform of Convolutions

Another useful property of the Fourier transform is that it turns convolutions
into products and vice versa. We shall first consider test functions. It
is easy to see that the product and convolution of functions in $S(\mathbb{R}^m)$
are again in $S(\mathbb{R}^m)$. Their behavior under Fourier transform is given by the
next lemma.

**Lemma 5.75.** Let $\phi, \psi \in S(\mathbb{R}^m)$. Then

$$
\mathcal{F}[\phi \ast \psi] = (2\pi)^{-m/2} \mathcal{F}[\phi] \mathcal{F}[\psi],
$$

(5.138)

$$
\mathcal{F}[\phi \psi] = (2\pi)^{-m/2} \mathcal{F}[\phi] \ast \mathcal{F}[\psi].
$$

(5.139)

**Proof.** We have

$$
\mathcal{F}[\phi \ast \psi](\xi) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\xi \cdot x} \int_{\mathbb{R}^m} \phi(x - y) \psi(y) \, dy \, dx
$$

$$
= (2\pi)^{-m/2} \int_{\mathbb{R}^m} \psi(y) \int_{\mathbb{R}^m} e^{-i\xi \cdot x} \phi(x - y) \, dx \, dy
$$

(5.140)

$$
= (2\pi)^{-m/2} \int_{\mathbb{R}^m} \psi(y) \int_{\mathbb{R}^m} e^{-i\xi \cdot (z + y)} \phi(z) \, dz \, dy
$$

$$
= (2\pi)^{m/2} \mathcal{F}[\phi](\xi) \mathcal{F}[\psi](\xi).
$$

This yields (5.138). Applying the inverse Fourier transform, we can restate
(5.138) as

$$
\mathcal{F}^{-1}[\hat{\phi} \hat{\psi}] = (2\pi)^{-m/2} \mathcal{F}^{-1}[\hat{\phi}] \ast \mathcal{F}^{-1}[\hat{\psi}].
$$

(5.141)

This and the trivial identity

$$
\mathcal{F}^{-1}[\hat{\phi}](x) = \mathcal{F}[\hat{\phi}](x)
$$

(5.142)

lead to (5.139).

We shall now extend this result to distributions.

**Theorem 5.76.** Let $f, g \in S'(\mathbb{R}^m)$ and let $g$ have compact support. Then
$f \ast g \in S'(\mathbb{R}^m)$ and

$$
\mathcal{F}[f \ast g] = (2\pi)^{m/2} \mathcal{F}[f] \mathcal{F}[g].
$$

(5.143)
5.4.5 Laplace Transforms

Let \( f \in \mathcal{S}'(\mathbb{R}) \) have support contained in \( \{ x \geq 0 \} \). Then obviously \( e^{-\mu x} f(x) \) is also in \( \mathcal{S}'(\mathbb{R}) \) for every \( \mu > 0 \). Formally, we have

\[
\mathcal{F}[e^{-\mu x} f](\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)e^{-i\xi x}e^{-\mu x} \, dx = \mathcal{F}[f](\xi - i\mu). \tag{5.147}
\]

Hence it is sensible to define \( \mathcal{F}[f](\xi - i\mu) \) as \( \mathcal{F}[f \exp(-\mu x)](\xi) \). This defines \( \mathcal{F}[f] \) in the lower half of the complex \( \xi \)-plane — as a generalized function of \( \text{Re} \, \xi \) depending on \( \text{Im} \, \xi \) as a parameter. Actually, however, \( \mathcal{F}[f] \) is an analytic function of \( \xi \) in the open half-plane \( \{ \text{Im} \, \xi < 0 \} \); this is shown by an argument similar to Example 5.70 (see Problem 5.52).
The Laplace transform is defined as
\[ \mathcal{L}[f](s) := \sqrt{2\pi} \mathcal{F}[f](-is); \] (5.148)
for \( f \in \mathcal{S}'(\mathbb{R}) \) with support in \( \{x \geq 0\} \) it is defined in the right half-plane. Formally, we have
\[ \mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x) \, dx. \] (5.149)
If \( f \) is not in \( \mathcal{S}'(\mathbb{R}) \), but \( \exp(-\mu x)f \) is in \( \mathcal{S}'(\mathbb{R}) \) for some positive \( \mu \), then we can define \( \mathcal{L}[f] \) in the half-plane \( \{\text{Re } s \geq \mu\} \). We note that by inverting the Fourier transform in (5.148) we obtain
\[ e^{-\mu x} f(x) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}[\mathcal{L}[f](\mu + i\xi)] \] (5.150)
or, equivalently,
\[ f(x) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{sx} \mathcal{L}[f](s) \, ds. \] (5.151)
In using (5.151), care must be taken that the resulting expression really vanishes for \( x < 0 \), since this was our basic assumption. Typically, one shows this by closing the contour of integration by a half-circle to the right; \( e^{sx} \) decays rapidly in the right half-plane. For this argument to work, it is necessary to choose \( \mu \) to the right of any singularities of \( \mathcal{L}[f] \).

We now give a few examples of applications of Laplace transforms.

**Example 5.77.** Consider the initial-value problem
\[ y'(x) = ay(x) + f(x), \quad y(0) = \alpha. \] (5.152)
We are interested in a solution for positive \( x \). For negative \( x \), we extend \( y \) and \( f \) by zero. The extended function does not satisfy (5.152); since it jumps from 0 to \( \alpha \) at the origin, its derivative contains a contribution \( \alpha \delta(x) \). Thus the proper equation for the extended functions is
\[ y'(x) = ay(x) + f(x) + \alpha \delta(x). \] (5.153)

We now take Laplace transforms. We obtain
\[ s \mathcal{L}[y](s) = a \mathcal{L}[y](s) + \mathcal{L}[f](s) + \alpha, \] (5.154)
and hence
\[ \mathcal{L}[y](s) = \frac{\mathcal{L}[f](s) + \alpha}{s - a}. \] (5.155)

To obtain \( y(x) \), we must now invert the Laplace transform; for instance, the inverse Laplace transform of \( 1/(s - a) \) is found from (5.151) as
\[ \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \frac{e^{sx}}{s - a} \, ds. \] (5.156)

This integral is easily evaluated by the method of residues; for \( \mu > a \), we obtain \( e^{ax} \) for \( x > 0 \) and 0 for \( x < 0 \). (Note that if we choose \( \mu < a \), we still get a solution of (5.153), but one that vanishes for \( x > 0 \) rather than \( x < 0 \); thus we do not get a solution of the original problem (5.152).) If we exploit the fact that the transform of a product is a convolution, we can now write the solution as

\[ y(x) = \alpha e^{ax} + \int_0^x e^{a(x-t)} f(t) \, dt, \quad x > 0; \quad (5.157) \]

of course we could have found this without using transforms.

**Example 5.78.** Abel’s integral equation is

\[ \int_0^x \frac{y(t)}{\sqrt{x-t}} \, dt = \sqrt{\pi} f(x), \quad (5.158) \]

again we seek a solution for \( x > 0 \) and we think of \( y \) and \( f \) as being extended by zero for negative \( x \). In order to have a solution, we must obviously have \( f(0) = 0 \). The left-hand side is the convolution of \( y \) and \( x^{-1/2} \), and the Laplace transform of a convolution is the product of the Laplace transforms. To find the transform of \( x^{-1/2} \), we compute

\[ \int_0^\infty e^{-sx} x^{-1/2} \, dx = \frac{1}{\sqrt{s}} \int_0^\infty e^{-t} t^{-1/2} \, dt = \sqrt{\frac{\pi}{s}} \quad (5.159) \]

for any real positive \( s \) and because of the uniqueness of analytic continuation this also holds for complex \( s \). Hence the transformed equation reads

\[ \mathcal{L}[y](s) = \sqrt{s} \mathcal{L}[f](s), \quad (5.160) \]

which we write as

\[ \mathcal{L}[y](s) = \frac{s \mathcal{L}[f](s)}{\sqrt{s}}. \quad (5.161) \]

Transforming back, we find

\[ y(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{f'(t)}{\sqrt{x-t}} \, dt. \quad (5.162) \]

**Example 5.79.** The Laplace transform is also applicable to initial-value problems for PDEs. We first remark that Definition 5.66 is easily generalized to define the Fourier transform of a generalized function with respect to only a subset of the variables. For example, when dealing with an initial-value problem, we can take the Laplace transform with respect to time. Of course, to make sense of boundary conditions, one needs to know more about the solution than that it is a generalized function. For example, in the following problem, we may think of \( u \) as a generalized function of \( t \) depending on \( x \) as a parameter.
We consider the initial/boundary-value problem
\[ u_t = u_{xx}, \quad x \in (0, 1), \ t > 0, \]
\[ u(x, 0) = 0, \quad x \in (0, 1), \]
\[ u(0, t) = u(1, t) = 1, \quad t > 0. \] (5.163)

As usual, we extend \( u \) by zero for negative \( t \). Laplace transform in time leads to the problem
\[ s \mathcal{L}[u](x, s) = \mathcal{L}[u_{xx}(x, s)], \]
\[ \mathcal{L}[u](0, s) = \mathcal{L}[u](1, s) = \frac{1}{s}. \] (5.164)

This equation has the solution
\[ \mathcal{L}[u](x, s) = \frac{\cosh(\sqrt{s}(x - 1/2))}{s \cosh(\sqrt{s}/2)}. \] (5.165)

The formula for the inverse transform yields
\[ u(x, t) = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{st} \frac{\cosh(\sqrt{s}(x - 1/2))}{s \cosh(\sqrt{s}/2)} \, ds. \] (5.166)

Here we can take \( \mu \) to be any positive number. The integral cannot be evaluated in closed form, but of course it can be evaluated numerically; it can also be used to deduce qualitative properties of the solution such as its regularity or its asymptotic behavior as \( t \to \infty \).

Problems

5.45. Let \( f \in \mathcal{D}(\mathbb{R}^m) \). Under what conditions is \( \mathcal{F}[f] \) also in \( \mathcal{D}(\mathbb{R}^m) \)? Hint: Consider \( \mathcal{F}[f] \) as a function of a complex argument.

5.46. Find the Fourier transforms of the following functions: \( \exp(-x^2) \), \( \frac{1}{1 + x^2} \), \( \frac{\sin x}{1 + x^2} \).

5.47. Check that (5.103), (5.104) and (5.110) hold for tempered distributions.

5.48. Find the Fourier transforms of \( |x| \), \( \sin(x^2) \), \( x_+^{1/2} \). Hint: Try modifying the functions using factors like \( e^{-\epsilon x^2} \) and passing to the limit.

5.49. Let \( \mathbf{A} \) be a nonsingular matrix. How is the Fourier transform of \( f(\mathbf{A}x) \) related to that of \( f(x) \)? Use the result to show that the Fourier transform of a radially symmetric tempered distribution is radially symmetric.

5.50. Use the Fourier transform to find the fundamental solution for the heat equation.

5.51. Use the Fourier transform to find the fundamental solution for Laplace’s equation in \( \mathbb{R}^3 \).
5.52. (a) Let \( f \in S'(\mathbb{R}) \) and assume that the support of \( f \) is contained in \( \{ x \geq 0 \} \). Show that the Fourier transform of \( f \) is an analytic function in the half-plane \( \text{Im } \xi < 0 \).
(b) Let \( f, g \in S'(\mathbb{R}) \) have support in \( \{ x \geq 0 \} \). Show that \( f \ast g \) also has support in \( \{ x \geq 0 \} \) and that (5.138) holds (in the pointwise sense) in the lower half-plane.

5.53. Use the Laplace transform to find the fundamental solution of the heat equation in one space dimension.

5.54. Show that, for any \( t > 0 \), (5.166) represents a \( C^\infty \) function of \( x \) for \( x \in [0, 1] \). Hint: First deform the contour into the left half-plane. Then differentiate under the integral.

5.55. In (5.163), replace the heat equation by the backwards heat equation \( u_t = -u_{xx} \). Explain what goes wrong when you try to solve the problem by Laplace transform.

5.5 Green’s Functions

In the previous two sections, we have considered fundamental solutions for PDEs with constant coefficients. Such fundamental solutions allow the solution of problems of the form \( L(D)u = f \), posed on all of space. In practical applications, however, one does not usually want to solve problems posed on all of space; rather one wants to solve PDEs on some domain, subject to certain conditions on the boundary. Green’s functions are the analogue of fundamental solutions for this situation. They can be found explicitly only in very special cases. Nevertheless, the concept of Green’s functions is useful for theoretical investigations. At present, we do not have the methods available to discuss the existence, uniqueness and regularity of Green’s functions for PDEs, and the discussions in this section will to a large extent be formal.

5.5.1 Boundary-Value Problems and their Adjoints

Definition 5.80. Let
\[
L(x, D) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha
\]
be a differential operator defined on \( \Omega \). Then the formal adjoint of \( L \) is the operator given by
\[
L^*(x, D)u = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha(a_\alpha(x)u(x)).
\]
The importance of this definition lies in the fact that

\[ (\phi, L(x, D)\psi) = (L^*(x, D)\phi, \psi) \]  

(5.169)

for every \( \phi, \psi \in D(\Omega) \). If the assumption of compact support is removed, then (5.169) no longer holds; instead the integration by parts yields additional terms involving integrals over the boundary \( \partial \Omega \). However, these boundary terms vanish if \( \phi \) and \( \psi \) satisfy certain restrictions on the boundary. We are interested in the case where the order of \( L \) is even, \( k = 2p \) and \( p \) linear homogeneous boundary conditions on \( \psi \) are given. It is then natural to seek \( p \) boundary conditions to be imposed on \( \phi \) which would make (5.169) hold. This leads to the notion of an adjoint boundary-value problem.

To make this idea concrete, let us first consider the case of ordinary differential equations. Let

\[ L(x, D)u = \sum_{i=0}^{2p} a_i(x) \frac{d^i u}{dx^i}(x), \quad x \in (a, b). \]  

(5.170)

We assume that \( a_i \in C^i([a, b]) \); this guarantees that the coefficients of \( L^* \) are continuous. Moreover, we assume that \( a_{2p}(x) \neq 0 \) for \( x \in [a, b] \). For any functions \( \phi, \psi \in C^{2p}[a, b] \), we compute

\[ (\phi, L(x, D)\psi) - (L^*(x, D)\phi, \psi) \]

\[ = \sum_{i=1}^{2p} \sum_{k=0}^{i-1} (-1)^k D^{i-k-1}\psi(b)D^k(a_i\phi)(b) \]

\[- \sum_{i=1}^{2p} \sum_{k=0}^{i-1} (-1)^k D^{i-k-1}\psi(a)D^k(a_i\phi)(a). \]  

(5.171)

The boundary terms can be recast in the form

\[ \sum_{k,l=1}^{2p} A_{kl}(b)D^{k-1}\phi(b)D^{l-1}\psi(b) - \sum_{k,l=1}^{2p} A_{kl}(a)D^{k-1}\phi(a)D^{l-1}\psi(a). \]  

(5.172)

Here \( A_{kl} \) vanishes for \( k + l > 2p + 1 \), and \( A_{k(2p+1-k)} = (-1)^{k-1}a_{2p} \). Since \( a_{2p} \) was assumed nonzero, this implies that the matrix \( A \) is nonsingular. Now assume that at the point \( b \) we have \( p \) linearly independent boundary conditions

\[ \sum_{j=1}^{2p} b_{ij}D^{j-1}\psi(b) = 0, \quad i = 1, \ldots, p. \]  

(5.173)

Let \( \mathbf{u} \) denote the \( 2p \) vector with components \( u_i = D^{i-1}\psi(b) \) and let \( \mathbf{v} \) denote the \( 2p \) vector with components \( v_i = D^{i-1}\phi(b) \). Then (5.173) constrains \( \mathbf{u} \) to a \( p \)-dimensional subspace \( X \) of \( \mathbb{R}^{2p} \). The image of \( X \) under \( \mathbf{A}(b) \) is a \( p \)-dimensional subspace \( Y \) of \( \mathbb{R}^{2p} \). In order to make the first term
in (5.172) vanish for every \( \psi \) that satisfies (5.173), it is necessary and sufficient to have \( v \) in the orthogonal complement of \( Y \). This yields a set of \( p \) boundary conditions for \( \phi \), which we call the adjoint boundary conditions. An analogous consideration applies at the point \( a \).

As an example, let \( L \psi = \psi^{'''} \) with boundary conditions \( \psi + \psi' = 2 \psi + \psi'' = 0 \) at each endpoint. In this case, \( L^* = L \), and (5.171) specializes to

\[
\int_a^b \phi(x) \psi^{'''}(x) \, dx = \int_a^b \phi^{'''}(x) \psi(x) \, dx \\
+ \phi'(b)\psi''(b) - \phi'(b)\psi''(b) + \phi''(b)\psi'(b) - \phi'''(b)\psi(b) \\
- \phi'(a)\psi''(a) + \phi'(a)\psi''(a) - \phi''(a)\psi'(a) + \phi'''(a)\psi(a).
\]

(5.174)
The matrix \( A \) in (5.172) is

\[
A = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\]

(5.175)
and the vector \( u \) is subject to the conditions \( u_1 + u_2 = 2u_1 + u_3 = 0 \).

A basis for the space \( X \) of vectors satisfying these conditions is given by the vectors \((1, -1, -2, 0)\) and \((0, 0, 0, 1)\). The images of these vectors under \( A \) are \((0, 2, -1, -1)\) and \((1, 0, 0, 0)\). Thus the vector \( v \) has to satisfy the conditions \( 2v_2 - v_3 - v_4 = 0 \), \( v_1 = 0 \), i.e., the adjoint boundary conditions are \( \phi = 2\phi' - \phi'' - \phi''' = 0 \).

Let now \( \Omega \) be a bounded domain in \( \mathbb{R}^m \) with a smooth boundary.\(^5\) Let \( L(x, D) \) be a differential operator of order \( 2p \) with smooth coefficients defined on \( \overline{\Omega} \). Moreover, let \( B_j(x, D) \), \( j = 1, \ldots, p \), be differential operators of orders less than \( 2p \) which are defined for \( x \in \partial \Omega \). In the following, we are concerned with the boundary-value problem

\[
L(x, D)u = f(x), \quad x \in \Omega, \\
B_j(x, D)u = 0, \quad x \in \partial \Omega, \quad j = 1, \ldots, p.
\]

(5.176)
We assume that there are additional differential operators

\[
S_j(x, D), \quad T_j(x, D), \quad C_j(x, D), \quad j = 1, \ldots, p,
\]
defined for \( x \in \partial \Omega \), with the following properties:

1. \( S_j, T_j \) and \( C_j \) have smooth coefficients and orders less than \( 2p \).

2. Given any set of smooth functions \( \phi_j, j = 1, \ldots, 2p \), defined on \( \partial \Omega \), there exist functions \( u, v \in C^{2p}(\overline{\Omega}) \) such that on \( \partial \Omega \) we have \( B_j u = \)

\(^5\)Since this section focuses on introducing basic concepts without any precise statement of results, we shall be vague about smoothness assumptions. “Smooth” should therefore be interpreted to mean “as smooth as may be needed.”
\[ \phi_j, S_j u = \phi_{p+j}, j = 1, \ldots, p \text{, and, respectively, } C_j v = \phi_j, T_j v = \phi_{p+j}, j = 1, \ldots, p. \]

3. For any \( u, v \in C^2(\Omega) \), we have
\[
\int_\Omega v L(x, D) u - u L^*(x, D) v \, dx = \int_{\partial \Omega} \sum_{j=1}^p S_j(x, D) u C_j(x, D) v \\
- B_j(x, D) u T_j(x, D) v \, dS.
\]

(5.177)

Of course, the question of what assumptions a boundary-value problem must satisfy for such operators to exist is of crucial importance; we shall return to this issue later when we discuss elliptic boundary-value problems. For the moment, we simply take the existence of the \( S_j, T_j \) and \( C_j \) for granted.

**Definition 5.81.** Let the preceding assumptions hold. Then the boundary-value problem
\[
L^*(x, D)v = g(x), \quad x \in \Omega, \\
C_j(x, D)v = 0, \quad x \in \partial \Omega, \ j = 1, \ldots, p.
\]

(5.178)
is called **adjoint** to (5.176).

We note that if \( u \) and \( v \) satisfy (5.176) and (5.178), respectively, then, according to (5.177),
\[
\int_\Omega f v - g u \, dx = 0.
\]

(5.179)

We have made no claim that the operators \( C_j \) are unique, and indeed, even for ordinary differential equations, the adjoint boundary conditions are determined only up to linear recombination. We can, however, give an intrinsic characterization of the set of functions characterized by the conditions \( C_j v = 0 \).

**Lemma 5.82.** Let \( v \in C^{2p}(\overline{\Omega}) \) and let \( X_B \) denote the set of all \( u \in C^{2p}(\overline{\Omega}) \) such that \( B_j u = 0 \) on \( \partial \Omega \) for \( j = 1, \ldots, p \). Then \( (v, Lu) = (L^* v, u) \) for every \( u \in X_B \) iff \( C_j v = 0 \) for \( j = 1, \ldots, p \).

**Proof.** One direction is obvious from (5.177). To see the converse, we note that by assumption we can construct \( u \in C^{2p}(\overline{\Omega}) \) such that \( B_j u = 0 \) and \( S_j u = \phi_j \), where \( \phi_j \) are given smooth functions. If \( (v, Lu) = (L^* v, u) \), then (5.177) assumes the form
\[
\int_{\partial \Omega} \sum_{j=1}^p \phi_j(x) C_j(x, D) v \, dS = 0.
\]

(5.180)

If this holds for arbitrary \( \phi_j \), then clearly \( C_j v \) must be zero. \( \square \)
Thus, although there may be different sets of adjoint boundary conditions, they must be equivalent to each other. As a caution, we note that equivalent sets of boundary conditions need not be linear combinations of each other. For example, let \( \partial \Omega \) be a closed curve in \( \mathbb{R}^2 \) and let \( s \) denote arclength. Then the conditions \( v = 0 \) and \( dv/ds + v = 0 \) are equivalent, although they are not multiples of each other.

We conclude this subsection with two examples:

**Example 5.83.** We have

\[
\int_{\Omega} u \Delta v - v \Delta u \, dx = \int_{\partial \Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, dS. \tag{5.181}
\]

Hence the Dirichlet and Neumann boundary-value problems for Laplace’s equation are their own adjoints.

**Example 5.84.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) bounded by a closed smooth curve. Let \( s \) denote arclength along the curve. Consider the boundary-value problem

\[
\Delta u = f(x), \ x \in \Omega, \quad \frac{\partial u}{\partial n} + \frac{\partial u}{\partial s} = 0, \ x \in \partial \Omega. \tag{5.182}
\]

We find

\[
\int_{\Omega} u \Delta v - v \Delta u \, dx = \int_{\partial \Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, ds
\]

\[
= \int_{\partial \Omega} u \left( \frac{\partial v}{\partial n} - \frac{\partial v}{\partial s} \right) - v \left( \frac{\partial u}{\partial n} + \frac{\partial u}{\partial s} \right) \, ds. \tag{5.183}
\]

Hence the adjoint boundary-value problem is

\[
\Delta v = g(x), \ x \in \Omega, \quad \frac{\partial v}{\partial n} - \frac{\partial v}{\partial s} = 0, \ x \in \partial \Omega. \tag{5.184}
\]

### 5.5.2 Green’s Functions for Boundary-Value Problems

We shall consider the boundary-value problem (5.176), and we make the assumptions of the last section.

**Definition 5.85.** A **Green’s function** \( G(x,y) \) for (5.176) is a solution of the problem

\[
L(x,D_x)G(x,y) = \delta(x - y), \quad x, y \in \Omega,
\]

\[
B_j(x,D_x)G(x,y) = 0, \quad x \in \partial \Omega, \ y \in \Omega, \ j = 1, \ldots, p. \tag{5.185}
\]

The first equation in (5.185) is to be interpreted in the sense of distributions. Of course, giving a meaning to the boundary conditions requires more smoothness of \( G \) than that it be a distribution. For elliptic boundary-value problems, however, it turns out that \( G \) is smooth as long as \( x \neq y \), and hence the interpretation of the boundary conditions poses no problems. Clearly, the concept of a Green’s function generalizes that of a
fundamental solution. If \( L \) has constant coefficients, it is in fact often advantageous to think of the Green’s function as a perturbation of the fundamental solution. Namely, if \( G(x - y) \) is the fundamental solution, we set \( G(x, y) = G(x - y) + g(x, y) \) where \( g \) satisfies

\[
L(x, D_x)g(x, y) = 0, \quad x, y \in \Omega, \tag{5.186}
\]

and for \( j = 1, \ldots, p \)

\[
B_j(x, D_x)g(x, y) = -B_j(x, D_x)G(x - y), \quad x \in \partial\Omega, \ y \in \Omega. \tag{5.187}
\]

If \( G \) is smooth for \( x \neq y \), then the right-hand side of (5.187) is smooth for every \( y \in \Omega \). For elliptic problems, we shall see in Chapter 9 that this implies that \( g \) is also smooth. In the interior of \( \Omega \), the fundamental solution in a sense “dominates” the Green’s function by contributing the most singular part.

It is of course of fundamental importance to identify classes of boundary-value problems for which Green’s functions exist and are unique. At present, we do not have the techniques available which are required to address this question, but we shall address the issue of existence for elliptic equations in Chapter 9.

If a Green’s function exists, then a formal solution of (5.176) is

\[
u(x) = \int_{\Omega} G(x, y)f(y) \, dy; \tag{5.188}\]

in fact, if \( f \in \mathcal{D}(\Omega) \), then (5.188) gives a solution of (5.176) under fairly minimal assumptions on \( G \). It suffices, for example, if \( G(\cdot, y) \) as an element of \( \mathcal{D}'(\Omega) \) depends continuously on \( y \) and \( G \) is smooth for \( x \neq y \). In particular, if the boundary-value problem (5.176) is uniquely solvable, (5.188) leads to the identity

\[
u(x) = \int_{\Omega} G(x, y)L(y, D_y)u(y) \, dy \tag{5.189}\]

for all \( u \in \mathcal{D}(\Omega) \). We shall now assume that \( G \) has sufficient regularity to establish (5.188) not only for \( u \in \mathcal{D}(\Omega) \), but for every \( u \in X_B \). Using (5.177), we conclude that

\[
u(x) = \int_{\Omega} u(y)L^*(y, D_y)G(x, y) \, dy + \int_{\partial\Omega} \sum_{j=1}^{p} S_j(y, D_y)u(y)C_j(y, D_y)G(x, y) \, dS_y. \tag{5.190}\]

If this holds for arbitrary \( u \in X_B \), we find that, for every \( x \in \Omega \), we must have

\[
L^*(y, D_y)G(x, y) = \delta(x - y), \quad x, y \in \Omega,
\]

\[
C_j(y, D_y)G(x, y) = 0, \quad y \in \partial\Omega, \ x \in \Omega, \ j = 1, \ldots, p. \tag{5.191}\]
That is, \( G \), regarded as a function of \( y \) for fixed \( x \), satisfies the adjoint boundary-value problem.

Using (5.191) and setting \( v(y) = G(x, y) \) in (5.177), we find

\[
  u(x) = \int_{\Omega} G(x, y)L(y, D_y)u(y) \, dy \\
  + \int_{\partial\Omega} \sum_{j=1}^{p} T_j(y, D_y)G(x, y)B_j(y, D_y)u(y) \, dS_y.
\]

Thus, if the inhomogeneous boundary-value problem

\[
  L(x, D)u = f(x), \quad x \in \Omega, \\
  B_j(x, D)u = \phi_j(x), \quad x \in \partial\Omega, \; j = 1, \ldots, p
\]

has a solution, then the solution is represented by

\[
  u(x) = \int_{\Omega} G(x, y)f(y) \, dy \\
  + \int_{\partial\Omega} \sum_{j=1}^{p} T_j(y, D_y)G(x, y)\phi_j(y) \, dS_y.
\]

As a caution, we note that in justifying the integration by parts which leads to (5.192), it is important that \( x \in \Omega \) so that \( G(x, y) \) is smooth for \( y \in \partial\Omega \). In general, (5.192) does not represent \( u(x) \) for \( x \in \partial\Omega \).

For some simple equations in simple domains, Green’s functions can be given explicitly. As an example, we consider Laplace’s equation on the ball \( B_R \) of radius \( R \) with Dirichlet boundary conditions. In this case, the Green’s function can be constructed by what is known as the method of images. The fundamental solution \( G(|x - y|) \) can be thought of as the potential of a point charge located at the point \( y \). The idea is now to put a second point charge at the reflected point \( \bar{y} = R^2 y / |y|^2 \) in such a way that the potentials of the two charges cancel each other on the sphere \( |x| = R \). This leads to the Green’s function

\[
  G(x, y) = G(|x - y|) - G\left(\frac{|y|}{R} |x - \bar{y}|\right) \\
  = G(\sqrt{|x|^2 + |y|^2 - 2x \cdot y}) - G(\sqrt{|x||y|/R^2}) + R^2 - 2x \cdot y).
\]

If \( y = 0 \), we set \( G(x, y) = G(|x|) - G(R) \). If \( |x| = R \), then \( G(x, y) = 0 \); moreover, we compute

\[
  \Delta_x G(x, y) = \delta(x - y) - \frac{|y|^2}{R^2} \delta(x - \bar{y}),
\]

which agrees with \( \delta(x - y) \) if \( x \) and \( y \) are restricted to \( B_R \). Hence \( G \) is indeed a Green’s function for the Dirichlet problem. We see that \( G \) is symmetric in its arguments, reflecting the self-adjointness of the Dirichlet problem.
The solution of the Dirichlet problem
\[ \Delta u = f(x), \; x \in B_R, \quad u = \phi(x), \; x \in \partial B_R \] (5.197)
is represented by (5.194):
\[ u(x) = \int_{B_R} G(x, y) f(y) \, dy + \int_{\partial B_R} \frac{\partial}{\partial n_y} G(x, y) \phi(y) \, dS_y. \] (5.198)
Moreover, a direct calculation shows that
\[ \frac{\partial}{\partial n_y} G(x, y) = \frac{R^2 - |x|^2}{\Omega m R |x - y|^m} \] (5.199)
(cf. Problem 5.60). This leads to Poisson's formula, which we have already encountered in Section 4.2.

### 5.5.3 Boundary Integral Methods

Let \( \Omega \) be a bounded domain with a smooth boundary. We want to solve the problem
\[ \Delta u = 0, \; x \in \Omega, \quad u = \phi(x), \; x \in \partial \Omega. \] (5.200)
If we knew the Green's function, we would have the representation
\[ u(x) = \int_{\partial \Omega} \frac{\partial}{\partial n_y} G(x, y) \phi(y) \, dS_y. \] (5.201)

We make an ansatz analogous to (5.201), with the Green's function replaced by the fundamental solution of the Laplace equation
\[ u(x) = \int_{\partial \Omega} \frac{\partial}{\partial n_y} G(|x - y|) g(y) \, dS_y. \] (5.202)

Here the function \( g \) is unknown, and we are seeking an equation relating \( g \) to \( \phi \).

We note that for any \( g \in C(\partial \Omega) \), the function \( u \) given by (5.202) is harmonic in \( \Omega \); we can simply take the Laplacian with respect to \( x \) under the integral. To satisfy the boundary condition, we must have
\[ \phi(x) = \lim_{z \to x, z \in \Omega} \int_{\partial \Omega} \frac{\partial}{\partial n_y} G(|z - y|) g(y) \, dS_y \] (5.203)
for \( x \in \partial \Omega \); this is the desired equation relating \( g \) to \( \phi \). One cannot pass to the limit in (5.203) by simply substituting \( x \) for \( z \); although the integral exists for \( z \in \partial \Omega \), it is discontinuous there. Indeed, we shall show below that actually
\[ \lim_{z \to x, z \in \Omega} \int_{\partial \Omega} \frac{\partial}{\partial n_y} G(|z - y|) g(y) \, dS_y \]
\[ = \int_{\partial \Omega} \frac{\partial}{\partial n_y} G(|x - y|) g(y) \, dS_y + \frac{1}{2} g(x). \] (5.204)
Recall that a similar situation applies to Cauchy’s formula; if $C$ is a smooth closed curve in the plane and $f$ is analytic, then

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

equals $f(z)$ for $z$ inside $C$, 0 for $z$ outside $C$ and (in the sense of principal value) $f(z)/2$ on $C$. Inserting (5.204) in (5.203), we obtain

$$\phi(x) - \frac{1}{2} g(x) = \int_{\partial \Omega} \frac{\partial}{\partial n_y} G(|x - y|) g(y) \, dS_y. \quad (5.206)$$

We have thus replaced the partial differential equation (5.200) by the equivalent integral equation (5.206). This has two advantages. As we shall see in Chapter 9, it is fairly easy to develop an existence theory for integral equations such as (5.206). Moreover, a numerical approach based on (5.206) rather than (5.200) has the advantage of working with a problem in a lower space dimension, which translates into fewer gridpoints. Indeed, there is an extensive literature on “boundary-element methods” for Laplace’s equation as well as for the Stokes equation.

It remains to verify (5.204). Let $z$ be close to $\partial \Omega$, and let $x$ be the point on $\partial \Omega$ nearest to $z$; without loss of generality, we may choose the coordinate system in such a way that $x$ is the origin and the normal to $\partial \Omega$ is in the $m$th coordinate direction. Let $N$ be a neighborhood of the origin; we can then split up the right-hand side of (5.203) as follows:

$$\int_{\partial \Omega} \frac{\partial}{\partial n_y} G(|z - y|) g(y) \, dS_y = \int_{\partial \Omega \cap N} \frac{\partial}{\partial n_y} G(|z - y|) g(y) \, dS_y$$

$$+ \int_{\partial \Omega \setminus N} \frac{\partial}{\partial n_y} G(|z - y|) g(y) \, dS_y. \quad (5.207)$$

The second term is continuous at $z = 0$. For the first term, we choose $N$ small enough so that $\partial \Omega \cap N$ can be represented in the form $y_m = \phi(y_1, \ldots, y_{m-1})$; we set $u = (y_1, \ldots, y_{m-1})$. This leads to

$$n_y = (-\nabla \phi, 1)/\sqrt{1 + |\nabla \phi|^2}, \quad dS_y = \sqrt{1 + |\nabla \phi|^2} \, du. \quad (5.208)$$

We may choose $N$ in such a way that $\partial \Omega \cap N = \{(u, \phi(u)) \mid |u| < \epsilon\}$. The first term on the right-hand side of (5.207) now assumes the form

$$\int_{\{|u|<\epsilon\}} \nabla_y G(|z - (u, \phi(u))|) \cdot (-\nabla \phi(u), 1) g(u, \phi(u)) \, du$$

$$= \int_{\{|u|<\epsilon\}} \frac{-u \cdot \nabla \phi(u) + \phi(u) - z_m}{\Omega_m (\sqrt{|u|^2 + |\phi(u) - z_m|^2})^m} g(u, \phi(u)) \, du. \quad (5.209)$$

We note that if we set $z_m = 0$ in (5.209), then $-u \cdot \nabla \phi(u) + \phi(u)$ is of order $|u|^2$ as $u \to 0$, hence the integrand is of order $|u|^{-(m-2)}$, i.e., it is integrable. Although the integral exists for $z = 0$, we cannot take the
limit $z_m \to 0$ under the integral. We shall now consider this limit with the constraint that $z_m < 0$. The term which needs to be investigated is

$$-z_m \int_{\{|u|<\epsilon\}} \frac{1}{\Omega_m(\sqrt{|u|^2 + |\phi(u) - z_m|^2})^m} g(u, \phi(u)) \, du. \tag{5.210}$$

For small $|u|$ and $|z_m|$, one has

$$\frac{1}{\sqrt{|u|^2 + (\phi(u) - z_m)^2}^m} = \frac{1}{\sqrt{|u|^2 + z_m^2}^m} (1 + O(|z_m|) + O(|u|^2)), \tag{5.211}$$

and it is easily checked that only the leading contribution leads to a discontinuity in (5.210) as $z_m \to 0$. It thus remains to consider the integral

$$-z_m \int_{\{|u|<\epsilon\}} \frac{1}{\Omega_m(\sqrt{|u|^2 + z_m^2}^m)} g(u, \phi(u)) \, du. \tag{5.212}$$

We define

$$I_r(g) = \frac{1}{r^m - 2\Omega_{m-1}} \int_{\{|u|=r\}} g(u, \phi(u)) \, dS. \tag{5.213}$$

We substitute $u = -z_m v$ in (5.212). This leads to the expression

$$\int_{\{|v|\leq-\epsilon/z_m\}} \frac{1}{\Omega_m(\sqrt{|v|^2 + 1})^m} g(-z_m v, \phi(-z_m v)) \, dv$$

$$= \frac{\Omega_{m-1}}{\Omega_m} \int_0^{-\epsilon/z_m} \frac{r^{m-2}}{(r^2 + 1)^{m/2}} I_{-z_m v}(g) \, dr. \tag{5.214}$$

In the limit $z_m \to 0^-$, we obtain

$$\frac{\Omega_{m-1}}{\Omega_m} g(0) \int_0^\infty \frac{r^{m-2}}{(r^2 + 1)^{m/2}} \, dr = \frac{1}{2} g(0). \tag{5.215}$$

Here we have used that

$$\int_0^\infty \frac{r^{m-2}}{(r^2 + 1)^{m/2}} \, dr = \frac{\Gamma(m-1)\sqrt{\pi}}{2\Gamma(m/2)} \tag{5.216}$$

(see [GR], p. 292) and the expression for $\Omega_m$ obtained in Problem 4.16.

Problems

5.56. On the interval $[0, 1]$, let $Lu = u'''' + u'$ with boundary conditions $u'' + u = u - u''''' = 0$ at the endpoints. Find the adjoint operator and the adjoint boundary conditions.

5.57. Find the Green’s function for the fourth derivative operator on $(0, 1)$ with boundary conditions $u(0) = u'(0) = u(1) = u'(1) = 0$.

5.58. Let $\Omega$ be a domain in $\mathbb{R}^2$ bounded by a smooth curve. Consider the equation $\Delta\Delta u = f$ with boundary conditions $\Delta u = \frac{\partial u}{\partial n} + \frac{\partial u}{\partial s} = 0$. Determine the adjoint boundary-value problem.