Center Manifold Theory
for Functional Differential Equations
of Mixed Type

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We are interested in nonlinear differential equations of the form

\[ \dot{x}(t) = G(x_t). \]

- \(x\) is a continuous function with \(x(t) \in \mathbb{R}\).
- \(x_t \in C([-1, 1])\) is the state of \(x\) at \(t\), i.e.,

\[ x_t(\theta) = x(t + \theta), \quad \theta \in [-1, 1]. \]

- \(G : C([-1, 1]) \rightarrow \mathbb{R}\) is sufficiently smooth.

Note that

- \(\dot{x}(t)\) depends on both past and future values of \(x\).
- We will look for \(x(t)\) near equilibria \(\bar{x}\), \(G(\bar{x}) = 0\).
Results

Starting point is the MFDE

\[ \dot{x}(t) = Lx_t + R(x_t). \]

- \( x_t \in X = C([-1,1]) \) is the state of \( x \) at \( t \).
- \( L : X \to \mathbb{R} \) is (for example) the linear operator

\[ \phi \mapsto A_0\phi(0) + A_-\phi(-1) + A_+\phi(+1). \]

- \( R : X \to \mathbb{R} \) is a nonlinear smooth operator with \( R(0) = 0 \) and \( DR(0) = 0 \).

Characteristic equation given by \( \Delta(z) = 0 \), with

\[ \Delta(z) = z - A_0 - A_-e^{-z} - A_+e^z. \]

We are specially interested in cases where there are eigenvalues on the imaginary axis, i.e., \( \Delta(i\omega) = 0 \) for some \( \omega \in \mathbb{R} \).
Recall

\[ \dot{x}(t) = Lx_t + R(x_t). \]  

(1)

As for delay equations, can define spectral projection
\( Q_0 : X \rightarrow X_0 \subset X \) onto finite dimensional subspace \( X_0 \) spanned by elements of the form

\[ \phi : t \mapsto t^l e^{i\omega t} \text{ with } \Delta(i\omega) = 0 \text{ and } \dot{\phi}(t) = L\phi_t. \]

Main result gives "smooth" \( u^* : X_0 \rightarrow \bigcap_{\eta > 0} BC_{1/\eta} \) such that

(i) Sufficiently small solutions \( x \) to (1) are captured via \( x = u^*(Q_0x_0) \).

(ii) Any \( \phi \in X_0 \) such that \( u^*\phi \) is sufficiently small, yields a solution \( x = u^*\phi \) to (1).

(iii) Dynamics on \( X_0 \) is captured by ODE (with \( A = L|_{X_0} \))

\[ \dot{\Phi}(t) = A\Phi(t) + f(\Phi(t)), \text{ where} \]

\[ f(\psi) = Q_0[L(u^*\psi - \psi)\theta + R((u^*\psi)\theta)]. \]
MFDE

Weird "interaction from future" often raises doubts about usefulness of MFDE in modelling applications.

• Lattice differential equations.

  One studies travelling wave solutions to infinite dimensional differential systems on discrete lattices.

  – Material science (crystals)
  – Image processing (recognizing edges / outlines in pictures)
  – Biology (signal propagation through nerves with discrete gaps)

• Solving optimal control problems with delays.

• Direct models that, indeed, contain past + future terms.
Modelling

Ferdinand Banks in 'Energy Economics: a modern introduction'

The difference between science and economics is that science aims at the understanding of the behaviour of nature, while economics is involved with an understanding of models— and many of these models have no relation to any state of nature that has ever existed on this planet [...] 

Our examples will come from economics.

- Optimal control capital dynamics with delay
- Direct life cycle model
Optimal Control Capital Market Dynamics

Consider an economy that starts at time $t = 0$. Total amount of capital in economy given by $k(t) \geq 0$. Investments given by $u(t)$.

Production takes time! (Rustichini, 1989)

Consumption $c(t)$ that is technologically feasible depends on investments and available capital, i.e.,

$$c(t) = C(u(t - \tau), k(t - \tau)).$$

Total welfare is given by $\ln c(t) > 0$ to ensure ”spreading out” of consumption.

Optimal control problem: maximize

$$\int_0^\infty e^{-\rho t} \ln C(u(t - \tau), k(t - \tau)) \, dt,$$

subject to $\dot{k}(t) = u(t - \tau) - gk(t - \tau)$.

Here $g$ is a form of capital decay rate and $\rho$ is the discount rate. This is a factor to correct for the fact that future welfare is rated to be less important than present welfare.
Euler Lagrange with Delays

Consider the problem to maximize the functional

\[ J(y) = \int_{0}^{\infty} f(t, y(t - \tau), y(t), \dot{y}(t - \tau), \dot{y}(t)) dt. \]

Introduce notation \( x(t) = y(t - \tau) \) and \( z(t) = y(t + \tau) \).

**Theorem 1 (Hughes 1968).** If \( y \) maximizes \( J \), then the following MFDE is satisfied

\[
\begin{align*}
D_3 f(t, x, y, \dot{x}, \dot{y}) & \quad + \quad D_2 f(t - \tau, y, z, \dot{y}, \dot{z}) \\
\quad = \quad & (d/dt)[D_5 f(t, x, y, \dot{x}, \dot{y})] \\
& \quad + \quad D_4 f(t + \tau, y, z, \dot{y}, \dot{z})
\end{align*}
\]

• Solving an optimal control problem with delays thus leads to a MFDE!
Market Dynamics with Delays

Application of Hughes’ result to maximize

\[ \int_0^\infty e^{-\rho t} \ln C(u(t - \tau), k(t - \tau)) \, dt, \]

leads to MFDE

\[ e^{-\rho(t+\tau)} [gD_1C/C + D_2C/C] (\dot{k}(t + \tau) + gk(t), k(t)) = \frac{d}{dt} \left( e^{-\rho t} D_1C/C (\dot{k}(t) + gk(t - \tau), k(t - \tau)) \right). \] (2)

Our example: \( C(u, k) = \sqrt{u - k} \). Steady state solution

\[ \bar{k} = \frac{e^{-2\rho \tau}}{4(\rho + ge^{-\rho \tau})^2}. \]

Linearizing around steady state and trying exponential solutions \( e^{zt} \) yields characteristic equation

\[ \Delta(z) = (z - \rho e^{-(z-\rho)\tau})(z - \rho + \rho e^{z\tau}) - \frac{1}{2}(\rho + ge^{-\rho \tau})(2\rho e^{\rho \tau} + g) = 0. \]
Market Dynamics with Delay

- Benhabib & Nishimura (1979) analyzed model without delays, but with at least three different economic goods.

- They found a pair of distinct eigenvalues that cross the imaginary axis when varying the parameters \((\rho, g)\), leading to Hopf bifurcation.

- Using CM reduction we are able to do the same for MFDE.

- \(\Delta(z)\) is transcendental function with infinitely many zeroes to the right and left of imaginary axis.

- Numerically study zeroes in a neighbourhood of origin.

- Economically interesting periodic solutions for models with only one production and one consumption good.
Zeroes

Roots of $\Delta(z)$ in rectangle $[-5, 5] \times [-23, 23]$, calculated using complex bisection.
Observe crossing of imaginary axis at $g \approx 13.667698$, $\rho = 0.80$, $\tau = 4$. 
Hopf bifurcation

Periodic solutions were computed for capital $k(t)$.

![Graph 1](image1.png)

- $g = 13.60$
- $g = 13.63$
- $g = 13.65$
- $g = 13.66$
- $g = 13.665$
- $g = 13.6673$

![Graph 2](image2.png)

- $g = 10.00$
- $g = 11.00$
- $g = 12.00$
- $g = 13.00$
- $g = 13.50$
Hopf bifurcation

Using computed periodic solutions, can construct local bifurcation diagram.

Explicit formula available for computing direction of bifurcation, derived by "lifting" finite dimensional Hopf bifurcation on CM.
Direct modelling: Life cycle model

Albis et al. (2004) consider a population model consisting of overlapping generations, that leads directly to MFDE.

- Fixed size population normalized to one. Each individual lives for time $T = 1$.

- Individuals born at time $s$ have assets $a(s, t)$ at time $t$.

- At birth, assets are zero, i.e., $a(s, s) = 0$.

- One does not die in debt, i.e., $a(s, s + 1) \geq 0$.

- Age-independent wages $w(t)$ are received.

- Interest rate $r(t)$.

- Individuals born at time $s$ consume $c(s, t)$ at time $t$.

Individual budget constraint:

$$\frac{\partial a(s, t)}{\partial t} = r(t)a(s, t) + w(t) - c(s, t).$$
Life cycle model II

Goal of every individual born at time $s$ is to maximize his lifetime welfare, given by

$$
\int_s^{s+1} \ln c(s, \tau) d\tau.
$$

Solving this optimization problem shows that the optimal asset distribution $a^*(s, t)$ depends on the interest rates and wages during the lifetime of an individual, i.e.,

$$
a^*(s, t) = F(r_{s+}, w_{s+}, t - s),
$$

for some $F$. Here $r_{s+} \in C([0, 1])$ is defined by $r_{s+}(\theta) = r(s + \theta)$.

The total amount of capital at any time $t$ is given by

$$
k(t) = \int_{t-1}^{t} a^*(\sigma, t) d\sigma,
$$

namely the total amount of assets owned by living individuals.
The economy has a single market good, that can be used for both production and consumption. It is produced at the rate $Q$ given by

$$Q(k(t), e(t), l(t)) = Ak(t)^\alpha (e(t)l(t))^{\beta}. $$

- $l(t)$ is the labour supply, in our case $l(t) = 1$.
- $e(t)$ accounts for the increase in labour efficiency over time.
- $A, \alpha, \beta > 0$ are parameters.
- $Q$ above is known as a Cobb-Douglas production function.

Note that interest rate $r(t)$ is, (by definition), the price of capital. Similarly, the wages $w(t)$ are the price of labour. They can be found by partial differentiation of $Q$.

$$r(t) = \frac{\partial Q}{\partial k} = \alpha Ak(t)^{\alpha-1} (e(t)l(t))^{\beta},$$

$$w(t) = \frac{\partial Q}{\partial l} = \beta Ak(t)^{\alpha} e(t)^{\beta} l(t)^{\beta-1}.$$
Life cycle model IV

We can now put everything back together.

\[ k(t) = \int_{t-1}^{t} a^*(\sigma, t) d\sigma, \]

\[ a^*(s, t) = F(r_{s+}, w_{s+}, t - s), \]

\[ r(t) = \frac{\partial Q}{\partial k} = \alpha A k(t)^{\alpha-1} (e(t) l(t))^{\beta}, \]

\[ w(t) = \frac{\partial Q}{\partial l} = \beta A k(t)^{\alpha} e(t)^{\beta} l(t)^{\beta-1}. \]

Substituting everything into the first equation, we arrive at

\[ k(t) = G(k_t, \alpha, \beta), \]

in which \( k_t \in C([-1, 1]) \) is given by \( k_t(\theta) = k(t + \theta). \)
Life cycle model V

Threefold differentiation of

\[ k(t) = G(k_t, \alpha, \beta). \]  \hspace{1cm} (3)

using the explicit form of \( G \) yields an MFDE

\[
\begin{align*}
\dddot{k}(t) &= f(k(t), k'(t), \dot{k}(t) - 1, k(t + 1), \int_t^{t+1} k(\tau)\alpha+\beta-1d\tau, \int_t^{t-1} k(\tau)\alpha+\beta-1d\tau).
\end{align*}
\]  \hspace{1cm} (4)

Albis et al. choose \( \alpha + \beta = 1 \), in which case the MFDE becomes linear.

We are interested in \( \alpha + \beta \neq 1 \), and we find that for every \( \gamma = \alpha + \beta \neq 1 \), (4) has a unique strictly positive equilibrium solution \( \bar{k} \).

Linearization around \( \bar{k} \) yields the characteristic equation with \( w = \alpha A \bar{k}^{\gamma-1} \),

\[
\Delta(z + w) = \alpha z^3 + w(\alpha - \gamma^2)z^2 - \gamma w^2(1 - \gamma)z - w\gamma((\gamma - 1)w^2 + 2\beta) + (z + w)^{-1}[2\beta w(\gamma z + w) \cosh z + w^2(\gamma - 1)(\gamma w^2 + 2\beta)].
\]
Life cycle model VI

Roots of $\Delta(z)$ with $\alpha = \beta$.

- Real root crosses the imaginary axis at $\alpha + \beta = 1$
- Apparently no imaginary pair of roots that cross imaginary axis.
- Since we demand $k > 0$, eigenmodes with $\text{Re} \ z > 0$ and $\text{Im} \ z \neq 0$ do not interest us.
- Loss of stability when $\alpha + \beta > 1$?
Recent Model (Albis et al. 2005)

Similar model, now with fixed wages only in the time period \([\alpha, 1-\alpha] \subset [0, 1]\) of an individual’s life.

Albis et al. attempt to find Hopf bifurcations for characteristic equation

\[
\Delta(z) = -\int_{\alpha}^{1-\alpha} \frac{e^{z\tau} d\tau}{1-2\alpha} + (1-\sigma) \int_{0}^{1} e^{z\tau} d\tau + \frac{\sigma}{\int_{0}^{1} e^{-z\tau} d\tau}.
\]

Fixed \(\alpha = 0.2\). Conditions for Hopf bifurcation not satisfied!
The program

• Analyze the linear homogeneous equation

\[ \dot{x}(\xi) = Lx_{\xi} \]

and determine set \( \mathcal{N}_0 \) of solutions that grow at most polynomially.

• Analyze the linear inhomogeneous equation

\[ \dot{x}(\xi) = Lx_{\xi} + f(\xi) \tag{5} \]

and find an "inverse" \( \mathcal{K} \) such that \( x = \mathcal{K}f \) solves (5) and \( \mathcal{K} \) projects out \( \mathcal{N}_0 \) in some sense.

• Analyze functional

\[ \mathcal{G}(u, y) = y + \mathcal{K}(R(u)), \]

for \( y \in \mathcal{N}_0 \). Use fixpoint arguments to find fixpoint \( \overline{u}(y) \) for \( \mathcal{G}(\cdot, y) \) for sufficiently small \( y \). Note that \( \overline{u}(y) \) solves

\[ \dot{x}(\xi) = Lx_{\xi} + R(x_{\xi}). \]
Comparison with delay equations

The homogeneous linear equation

\[ \dot{x}(\xi) = Lx_\xi, \]

is for general MFDE not a well-posed initial value problem, if we demand \( x \) continuous.

Example: \( x|_{[-1,1]} = 1 \) and

\[ x'(\xi) = x(\xi - 1) + x(\xi + 1). \]
Comparison with delay equations II

Linear delay equations (RFDE) however do allow unique forward continuation of solutions, i.e., the problem

\[ \dot{x}(\xi) = L_d x_{\xi-}, \quad x_{0-} = \phi, \]

in which \( x_{\xi-} \in C([-1, 0]) \) is defined by \( x_{\xi-}(\theta) = x(\xi + \theta) \), has solution

\[ x_{t-} = S(t)\phi, \]

in which \( S(t) \) is an eventually compact semigroup, with generator

\[ A : D(A) = \{ \phi \in C^1([-1, 0]) \mid \dot{\phi}(0) = L_d \phi \} \to C([-1, 0]) \]

\[ A\phi = \dot{\phi} \]

One can thus employ all the strong results from semigroup theory, in particular variation of constants formula!

As in study of elliptic PDEs (Mielke, Kirchgässner), need to construct pseudo inverse \( \mathcal{K} \) by hand for MFDE.
Linear inhomogeneous equations

The important step is to analyze the linear inhomogeneous equation

\[ \dot{x}(\xi) = Lx_{\xi} + f(\xi), \]  

(6)

for functions \( f : \mathbb{R} \rightarrow \mathbb{R} \).

Mallet-Paret established result for hyperbolic versions of (6), using Laplace transform techniques.

**Theorem 2.** Suppose that \( \Delta(z) = 0 \) has no roots on the imaginary axis. Then for every \( f \in L^\infty \), (6) has a unique solution in \( W^{1,\infty} \), given by

\[ x(\xi) = \int_{-\infty}^{\infty} G(\xi - s)f(s)ds, \]

where \( G \) has Fourier transform \( \hat{G}(\eta) = \Delta(i\eta)^{-1} \).
Nonhyperbolic inhomogeneous equations

However, when we have spectrum on the imaginary axis,

\[ \dot{x}(\xi) = Lx_\xi \]

has set of solutions \( \mathcal{N}_0 \) that are bounded or grow at most at a polynomial rate. Need to find a "pseudo-inverse" that projects out these solutions in some way.

Want to apply Mallet-Paret theorem, but need to "shift" the eigenvalues off the imaginary axis first.

This can be done by multiplying the equations with exponentials \( e^{\eta \xi} \).

Laplace transform enables us to link the "projecting out" of solutions in \( \mathcal{N}_0 \), to the projection \( Q_0 \) onto the spectral subspace \( X_0 \).
Dynamics on the Center Manifold

For any \( \phi \in X_0 \), define the function \( \Phi : \mathbb{R} \rightarrow X_0 \) by

\[
\Phi(t) = Q_0[(u^*\phi)_t].
\]

Then \( \Phi \) satisfies an ODE on the center manifold

\[
\dot{\Phi}(\xi) = A\Phi(\xi) + f(\Phi(\xi)), \text{ where }
\]

\[
f(\psi) = Q_0[L(u^*\psi - \psi)\theta + \tilde{R}((u^*\psi)\theta)].
\]

(7)

Here \( \tilde{R} \) is "cut-off" version of \( R \), i.e., globally bounded.

Conversely, if \( \Psi \) satisfies (7), then \( x = u^*\Psi(0) \) satisfies

\[
\dot{x}(t) = Lx_t + \tilde{R}(x_t)
\]

and in addition, \( x_t = (u^*\Psi(t))_0 \).

For delay eqs. (RFDE), variation of constants approach yields

\[
f_d(\psi) = Q_0[\tilde{R}((u^*\psi)_0)\delta(\theta)].
\]

Although this appears to differ from (7), examples indicate that our \( f \) restricted to RFDE yields same Taylor expansions as \( f_d \).
Further issues

- In the context of RFDE, one can also study invariant stable and unstable manifolds that capture sufficiently small solutions on the halflines $\mathbb{R}_\pm$.

- This feature is absent from our analysis here, due to the ill-posedness of the initial value problem. We cannot define a suitable solution operator for linear systems on half lines.

- Mallet - Paret and Verduyn Lunel (2001) have some results in this direction.

- Need to develop Floquet theory to analyze periodic solutions of MFDE.

- Need to do some work on stability analysis. Spectrum is unbounded to the left and right of imaginary axis. But perhaps results restricted to positive solutions? Useful for modelling applications.
References