Multi-Dimensional Stability of Travelling Waves through Rectangular Lattices

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( Joint work with E. van Vleck and A. Hoffman )
2d Lattice Differential Equation

Focus in this talk: lattice differential equation (LDE)

\[ \dot{u}_{i,j}(t) = [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a). \]

- Often called: discrete Nagumo equation.
- Two dimensional spatial lattice: \((i, j) \in \mathbb{Z}^2\).
- Nonlinearity \(g\) is \textbf{bistable}.
- Discrete Laplacian \(\Delta^+\) mixes nearest neighbours:

\[ [\Delta^+ u]_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}. \]
Recall the dynamics:

\[
\dot{u}_{i,j}(t) = [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).
\]

Bistable nonlinearity \( g \) given by

\[
g(u; a) = u(a - u)(u - 1).
\]

Two **stable** equilibria \( u = 0 \) and \( u = 1 \).

One **unstable** equilibrium \( u = a \).
Lattice equations: Travelling Waves

Recall the dynamics:

$$\dot{u}_{i,j}(t) = [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).$$

The nonlinearity $g$ 'pulls' $u$ towards either $u = 0$ or $u = 1$ [competition].

The discrete diffusion 'smooths' out any wrinkles in $u$.

Travelling waves: compromise between these two forces.

$$u_{i,j}(t) = \Phi((\cos \theta, \sin \theta) \cdot (i,j) + ct), \quad \Phi(-\infty) = 0, \quad \Phi(+\infty) = 1.$$
Lattice equations: Travelling Waves

Recall the dynamics:

\[ \dot{u}_{i,j}(t) = \left[ \Delta^+ u(t) \right]_{i,j} + g(u_{i,j}(t); a). \]

- Travelling waves connecting \( u \equiv 0 \) to \( u \equiv 1 \) must satisfy

\[ c\Phi'(\xi) = \Phi(\xi + \cos \theta) + \Phi(\xi - \cos \theta) + \Phi(\xi + \sin \theta) + \Phi(\xi - \sin \theta) - 4\Phi(\xi) + g(\Phi(\xi); a) \]

This is a mixed type functional differential equation (MFDE).

Direction \( \theta \) explicitly appears in wave equation.
Lattice equations: Travelling Waves

Recall the dynamics:

\[
\dot{u}_{i,j}(t) = [\Delta^+ u(t)]_{i,j} + g(u_{i,j}(t); a).
\]

Existence of travelling waves For each \( a \in (0, 1) \) and \( \theta \in [0, 2\pi] \) there exists a travelling wave.

Speed \( c(a, \theta) \) is unique.

If \( c \neq 0 \), then wave profile \( \Phi \) is unique and also monotone, i.e. \( \Phi' > 0 \).

[Mallet-Paret]

Dependence of \( c \) on angle \( \theta \) and detuning parameter \( a \) very delicate. [Aaron Hoffman’s talk]

In this talk: we fix \( (a, \theta) \) and assume that \( c \neq 0 \).

Goal: understand stability of the travelling wave.
Assumption: we have a wave solution \((c, \Phi)\) travelling \((c \neq 0)\) in rational direction \((\sigma_1, \sigma_2) \in \mathbb{Z}^2\).

Naive Ansatz

\[ u_{ij}(t) = \Phi(i\sigma_1 + j\sigma_2 + ct) + v_{ij}(t). \]

Need to understand behaviour of perturbation \(v(t)\).

First step: want natural coordinates parallel and perpendicular to propagation of wave.

\[
\begin{align*}
    n &= i\sigma_1 + j\sigma_2 & \text{parallel} \\
    l &= i\sigma_2 - j\sigma_1 & \text{transverse}.
\end{align*}
\]
Stability - Coordinate System

New coordinates:
\[ n = i \sigma_1 + j \sigma_2 \quad \text{parallel} \]
\[ l = i \sigma_2 - j \sigma_1 \quad \text{transverse}. \]

Old coordinates:
\[ i = [\sigma_1^2 + \sigma_2^2]^{-1}[n\sigma_1 + l\sigma_2] \]
\[ j = [\sigma_1^2 + \sigma_2^2]^{-1}[n\sigma_2 - l\sigma_1] \]

Equation only posed on sublattice of \((n, l) \in \mathbb{Z}^2\) in new coordinates.

Remember: \((\sigma_1, \sigma_2) \in \mathbb{Z}^2\).
Stability - Coordinate System

In new coordinates, LDE becomes

\[ \dot{u}_{nl}(t) = [\Delta^\times u(t)]_{nl} + g(u_{nl}(t)). \]

The discrete operator \( \Delta^\times \) now acts as

\[
[\Delta^\times u]_{n,l} = u_{n+\sigma_1,l+\sigma_2} + u_{n+\sigma_2,l-\sigma_1} + u_{n-\sigma_1,l-\sigma_2} + u_{n-\sigma_2,l+\sigma_1} - 4u_{n,l}.
\]

All geometrical information encoded in \( \Delta^\times \).

Travelling wave becomes: \( u_{nl}(t) = \Phi(n + ct) \)

Special cases \( (\sigma_1, \sigma_2) = (1, 0) \) or \( (0, 1) \) (horizontal or vertical waves): \( \Delta^\times = \Delta^+ \).
**Stability - Perturbation**

Substituting naive perturbation Ansatz

\[ u_{nl}(t) = \Phi(n + ct) + v_{nl}(t) \]

into LDE we obtain

\[ \dot{v}_{nl}(t) = [\Delta \times v(t)]_{nl} + g'(\Phi(n + ct))v_{nl}(t) + O(|v_{nl}(t)|^2). \]

(L) Need to understand growth rate of linear system

\[ \dot{v}_{nl}(t) = [\Delta \times v(t)]_{nl} + g'(\Phi(n + ct))v_{nl}(t). \]

In general, since we are in 2d, expect something algebraic.

(NL) Quadratic nonlinearities combined with slow algebraic decay spell trouble.

\[ \int_0^t (1 + t - t_0)^{-1/2} [(1 + t_0)^{-1/2}]^2 dt_0 \sim \ln(1 + t)(1 + t)^{-1/2}. \]
Focus on linear LDE posed on $\mathbb{Z}^2$:

$$\dot{v}_{nl}(t) = [\Delta^x v(t)]_{nl} + g'\left(\Phi(n + ct)\right) v_{nl}(t).$$

Observe: transverse coordinate $l$ does not appear in coefficients.

Ideal for Fourier transform in transverse direction.

Write, for $\omega \in [-\pi, \pi]$:

$$\hat{v}_n(\omega) = \sum_{l \in \mathbb{Z}} v_{nl} e^{-i\omega l}.$$

Inverse transformation:

$$v_{nl} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{il\omega} \hat{v}_n(\omega) \, d\omega.$$
Stability - Linear System

Focus on linear LDE posed on \( \mathbb{Z}^2 \):

\[
\dot{v}_{nl}(t) = [\Delta \times v(t)]_{nl} + g'(\Phi(n + ct))v_{nl}(t).
\]

Observe: transverse coordinate \( l \) does not appear in coefficients.

Ideal for Fourier transform in \textit{transverse} direction.

System is decoupled into

\[
\frac{d}{dt}\hat{v}_n(\omega, t) = [\hat{\Delta} \times (\omega)\hat{v}(\omega, t)]_n + g'(\Phi(n + ct))\hat{v}_n(\omega, t),
\]

with

\[
[\hat{\Delta} \times (\omega)v]_n = e^{+i\omega\sigma_2}v_{n+\sigma_1} + e^{-i\omega\sigma_1}v_{n+\sigma_2} + e^{-i\omega\sigma_2}v_{n-\sigma_1} + e^{i\omega\sigma_1}v_{n-\sigma_2} - 4v_n.
\]

In other words, for each frequency \( \omega \) we have an LDE posed on a 1d lattice (in \textit{parallel} coordinate \( n \)).
Recall decoupled LDE

\[
\frac{d}{dt} \hat{v}_n(\omega, t) = \left[ \hat{\Delta}^\times(\omega) \hat{v}(\omega, t) \right]_n + g'(\Phi(n + ct))\hat{v}_n(\omega, t),
\]

with

\[
\left[ \hat{\Delta}^\times(\omega)v \right]_n = e^{+i\omega\sigma_2}v_{n+\sigma_1} + e^{-i\omega\sigma_1}v_{n+\sigma_2} + e^{-i\omega\sigma_2}v_{n-\sigma_1} + e^{i\omega\sigma_1}v_{n-\sigma_2} - 4v_n.
\]

Special case \( \omega = 0 \). Write \( w_n(t) = \hat{v}_n(0, t) \). We get

\[
\frac{d}{dt} w_n(t) = \left[ \hat{\Delta}^\times(0) w(t) \right]_n + g'(\Phi(n + ct))w_n(t),
\]

with

\[
\left[ \hat{\Delta}^\times(0) w \right]_n = w_{n+\sigma_1} + w_{n+\sigma_2} + w_{n-\sigma_1} + w_{n-\sigma_2} - 4w_n.
\]
In special case $\omega = 0$, writing $w_n(t) = \hat{v}_n(0, t)$, we hence have:

$$\frac{d}{dt} w_n(t) = w_{n+\sigma_1}(t) + w_{n+\sigma_2}(t) + w_{n-\sigma_1}(t) + w_{n-\sigma_2}(t) - 4w_n(t) + g'(\Phi(n + ct))w_n(t).$$

Notice that $w_n(t) = \Phi'(n + ct)$ is a solution.

Indeed: wave profile $\Phi$ had to satisfy

$$c\Phi'(\xi) = \Phi(\xi + \sigma_1) + \Phi(\xi + \sigma_2) + \Phi(\xi - \sigma_1) + \Phi(\xi - \sigma_2) - 4\Phi(\xi) + g(\Phi(\xi)).$$

The zero-frequency component is hence the usual linearization around the travelling wave, just like in 1d.
Stability - 1d Linear Systems

Need to understand 1d LDE’s, e.g.

\[ \dot{U}_j(t) = U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t)), \quad j \in \mathbb{Z}. \]

Write as

\[ \dot{U}(t) = \mathcal{F}(U(t)), \]

with \( \mathcal{F} : \ell^\infty(\mathbb{Z}; \mathbb{R}) \rightarrow \ell^\infty(\mathbb{Z}; \mathbb{R}) \).

View as ODE posed on sequence space \( \ell^\infty(\mathbb{Z}; \mathbb{R}) \).

Suppose we have a wave solution \( U_j(t) = \Phi(j + ct) \) with \( c > 0 \), with

\[ \lim_{\xi \to -\infty} \Phi(\xi) = 0, \quad \lim_{\xi \to +\infty} \Phi(\xi) = 1. \]

Want to understand linear behaviour of \( U(t) = \bar{U}(t) + V(t) \).
Stability - 1d Linear Systems

Linear dynamics for $V(t) = U(t) - \overline{U}(t)$:

$$\dot{V}(t) = D\mathcal{F}(\overline{U}(t))V(t), \quad V(t) \in \ell^\infty(\mathbb{Z}; \mathbb{R}).$$

Problem: Non-Autonomous!

Remember: $U_j(t) = \Phi(j + ct)$. We DO have shift-periodicity

$$U_j(t + 1/c) = U_{j+1}(t) \quad (\Phi(j + 1)).$$
Stability - 1d Linear Systems

Linear behaviour $V(t) = U(t) - \bar{U}(t)$:

**Green’s function** $[\mathcal{G}(t, t_0)]_{jj_0}$ is value of $V_j(t)$ for unique solution to linearized LDE

\[
\begin{align*}
\dot{V}(t) &= DF(\bar{U}(t))V(t) \\
V_{j'}(t_0) &= \delta_{j',j_0}.
\end{align*}
\]
Stability - 1d Linear Systems

Linear behaviour \( V(t) = U(t) - \overline{U}(t) \):

**Green’s function** \([G(t, t_0)]_{j,j_0}\) is value of \( V_j(t) \) for unique solution to linearized LDE

\[
\begin{align*}
\dot{V}(t) &= D \mathcal{F}\left( \overline{U}(t) \right)V(t) \\
V_{j'}(t_0) &= \delta_{j',j_0}.
\end{align*}
\]

For \( V \in \ell^\infty(\mathbb{Z}; \mathbb{R}) \), write \( G(t, t_0)V \) for sequence

\[
[G(t, t_0)V]_j = \sum_{j_0 \in \mathbb{Z}} [G(t, t_0)]_{j,j_0}V_{j_0}
\]

(convolution).

All information (time + space) on linear system encoded in \( G(t, t_0) \).
Stability - 1d Linear Systems

To understand $G(t, t_0)$ must solve

$$\dot{V}(t) = D\mathcal{F}(\overline{U}(t))V(t).$$

[Chow, Mallet-Paret, Shen] Can exploit shift-periodicity to develop shift-periodic Floquet theory.

Problem: must analyze 'monodromy map' $G(t_0 + \frac{1}{c}, t_0)$ 'by hand'. Heavily dependent on ad-hoc arguments e.g. comparison principles. All arguments in sequence space $\ell^\infty(\mathbb{Z}; \mathbb{R})$.

Nevertheless, authors managed to understand discrete Nagumo equation.

Our goal: Make connection with highly developed nonlinear stability theory for PDEs [Zumbrun, Howard, ...].
Stability - 1d Linear Systems

Recall linear problem on $\ell^\infty(\mathbb{Z}; \mathbb{R})$:

$$\dot{V}(t) = D\mathcal{F}(\overline{U}(t))V(t),$$

which for discrete Nagumo LDE is:

$$\dot{V}_j(t) = V_{j+1}(t) + V_{j-1}(t) - 2V_j(t) + g'(\Phi(j + ct))V_j(t).$$

We 'fill in the gaps' between lattice points and look for solutions

$$V_j(t) = e^{\lambda t}v(j + ct).$$

Here $\lambda \in \mathbb{C}$ is spectral parameter and $v$ must be bounded and solve

$$cv'(\xi) + \lambda v(\xi) = v(\xi - 1) + v(\xi + 1) - 2v(\xi) + g'(\Phi(\xi))v(\xi)$$

in comoving frame $\xi = j + ct$. Write as $\mathcal{L}v = \lambda v$ with

$$[\mathcal{L}v](\xi) = -cv'(\xi) + v(\xi - 1) + v(\xi + 1) - 2v(\xi) + g'(\Phi(\xi))v(\xi).$$
**Fundamental relation**

Reminder: **Green’s function** $[\mathcal{G}(t, t_0)]_{jj_0}$ is value of $V_j(t)$ for unique solution to linearized LDE

$$\begin{align*}
\dot{V}(t) & = DF\left(\overline{U}(t)\right)V(t) \\
V_{j'}(t_0) & = \delta_{j', j_0}.
\end{align*}$$

**Thm.** [Benzoni-Gavage, Huot, Rousset] For $\gamma \gg 1$ and $t > t_0$,

$$[\mathcal{G}(t, t_0)]_{jj_0} = \frac{-1}{2\pi i} \int_{\gamma-i\pi c}^{\gamma+i\pi c} e^{\lambda(t-t_0)} G_\lambda(j + ct, j_0 + ct_0) d\lambda.$$ 

**Resolvent kernel** $G_\lambda(\xi, \xi_0)$ is unique solution [if defined] to

$$(\mathcal{L} - \lambda)G_\lambda(\cdot, \xi_0) = \delta(\xi - \xi_0).$$
Stability

Recall identity \((\gamma \gg 1 \text{ and } t > t_0)\)

\[
[G(t, t_0)]_{jj_0} = \frac{-1}{2\pi i} \int_{\gamma-i\pi c}^{\gamma+i\pi c} e^{\lambda (t-t_0)} G_\lambda (j + ct, j_0 + ct_0) d\lambda.
\]

Can view this as refined version of meta-identity

\[
e^{tL} = -\frac{1}{2\pi i} \int e^{\lambda t} [L - \lambda]^{-1} d\lambda.
\]

Have to worry about invertibility of \(L - \lambda\), i.e. study spectrum of \(L\).

For example \(L\Phi' = 0\) (translational invariance), so \(\lambda = 0\) in spectrum.
Recall identity ($\gamma \gg 1$ and $t > t_0$)

$$[\mathcal{G}(t, t_0)]_{j_0} = \frac{-1}{2\pi i} \int_{\gamma - i\pi c}^{\gamma + i\pi c} e^{\lambda(t-t_0)} G_\lambda(j + ct, j_0 + ct_0) d\lambda.$$
Recall identity (\( \gamma \gg 1 \) and \( t > t_0 \))

\[
\begin{align*}
[G(t, t_0)]_{jj_0} &= \frac{-1}{2\pi i} \int_{\gamma-i\pi c}^{\gamma+i\pi c} e^{\lambda(t-t_0)} G_\lambda(j + ct, j_0 + ct_0) d\lambda.
\end{align*}
\]

Main goal: construct expressions for \( G_\lambda(\xi, \xi_0) \) that can be extended meromorphically in \( \lambda \) near poles of \( [L - \lambda]^{-1} \).

Can do this if translational eigenvalue \( \lambda = 0 \) is a simple eigenvalue \([H. + Sandstede]\). In particular, if \( \text{Ker} L = \text{span}\{\Phi'\} \) and \( \Phi' \notin \text{Range} L \).

One obtains

\[
G_\lambda(\xi, \xi_0) = \lambda^{-1} \Phi'(\xi) \Psi(\xi_0) + O(e^{-\nu |\xi-\xi_0|}),
\]

where we have

\[
\text{Ker} L^* = \text{span}\{\Psi\},
\]

with \( L^* \) the formal adjoint of \( L \).
Stability - 1d Linear Systems

Recall identity (\( \gamma \gg 1 \) and \( t > t_0 \))

\[
[\mathcal{G}(t, t_0)]_{jj_0} = -\frac{1}{2\pi i} \int_{\gamma-i\pi c}^{\gamma+i\pi c} e^{\lambda(t-t_0)} G_{\lambda}(j + ct, j_0 + ct_0) d\lambda.
\]

Using meromorphic form

\[
G_{\lambda}(\xi, \xi_0) = \lambda^{-1} \Phi'(\xi) \Psi(\xi_0) + O(e^{-\nu|\xi-\xi_0|}),
\]

we now obtain the key result

\[
[\mathcal{G}(t, t_0)]_{jj_0} = \Phi(j + ct) \Psi(j_0 + ct_0) + O(e^{-\nu(t-t_0)}e^{-\nu|j+ct-j_0-ct_0|})
\]

In particular, Green’s function for 1d lattice system can be ’read-off’ from well-behaved spectral pictures.
Stability - back to 2d

Remember: for $\omega = 0$, writing $w_n(t) = \hat{v}_n(0, t)$, we had:

$$\frac{d}{dt} w_n(t) = w_{n+\sigma_1}(t) + w_{n+\sigma_2}(t) + w_{n-\sigma_1}(t) + w_{n-\sigma_2}(t) - 4w_n(t) + g'(\Phi(n + ct))w_n(t).$$

In this case, the relevant linear operator is:

$$[L_0 w](\xi) = -cw'(\xi) + w(\xi \pm \sigma_1) + w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

Remember $L_0\Phi' = 0$. We also have: $L_0^*\Psi = 0$ for the adjoint $\Psi$ which has $\Psi(\xi) > 0$ [Mallet-Paret].

For the Green’s function we hence get

$$[G_{\omega=0}(t, t_0)]_{nn_0} = \Phi'(n + ct)\Psi(n_0 + ct_0) + O(e^{-\nu(t-t_0)}e^{-\nu|n+ct-n_0-ct_0|}).$$

Note: no temporal decay.
Stability - Linear System

Back to $\omega \neq 0$. Recall decoupled LDE

$$\frac{d}{dt} \hat{v}_n(\omega, t) = [\hat{\Delta}^\times(\omega) \hat{v}(\omega, t)]_n + g'(\Phi(n + ct))\hat{v}_n(\omega, t),$$

with

$$[\hat{\Delta}^\times(\omega)v]_n = e^{+i\omega\sigma_2}v_{n+\sigma_1} + e^{-i\omega\sigma_1}v_{n+\sigma_2} + e^{-i\omega\sigma_2}v_{n-\sigma_1} + e^{i\omega\sigma_1}v_{n-\sigma_2} - 4v_n.$$

Relevant operator now is:

$$[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega\sigma_2}w(\xi \pm \sigma_1) + e^{\mp i\omega\sigma_1}w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

Need to understand spectrum of this operator.

What happens to zero eigenvalue for $\omega \approx 0$?
Stability - Linear System

Recall \( \omega \)-dependent linear operators

\[
[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega \sigma_2} w(\xi \pm \sigma_1) + e^{\mp i\omega \sigma_1} w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).
\]

There exists a branch

\[
\omega \mapsto (\lambda_\omega, \phi_\omega, \psi_\omega)
\]

for \( \omega \approx 0 \) with

\[
[\mathcal{L}_\omega - \lambda_\omega] \phi_\omega = 0, \quad [\mathcal{L}_\omega^* - \lambda_\omega^*] \psi_\omega = 0
\]

Of course, \( \lambda_0 = 0, \phi_0 = \Phi' \) and \( \psi_0 = \Psi \).

Key assumption:

\[
\text{Re} \lambda_\omega \leq -\kappa \omega^2, \quad \omega \approx 0, \quad \kappa > 0
\]

For general directions \( (\sigma_1, \sigma_2) \in \mathbb{Z}^2 \), can only establish this with numerics.
Stability - Linear System

Recall $\omega$-dependent linear operators

$$[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + e^{\pm i\omega \sigma_2} w(\xi \pm \sigma_1) + e^{\mp i\omega \sigma_1} w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).$$

In special case $(\sigma_1, \sigma_2) = (1, 0)$ we get

$$[\mathcal{L}_\omega w](\xi) = -cw'(\xi) + w(\xi \pm 1) + 2\cos\omega w(\xi) - 4w(\xi) + g'(\Phi(\xi))w(\xi)$$
$$= [\mathcal{L}_0 w](\xi) + 2(\cos\omega - 1)w(\xi).$$

This immediately gives $\lambda_\omega = 2(\cos\omega - 1)$ and $\phi_\omega = \Phi'$. Eigenfunctions $\phi_\omega$ now independent of $\omega$. 
Stability - Linear System

Recall $\omega$-dependent linear operators

\[
\mathcal{L}_\omega w(\xi) = -cw'(\xi) + e^{\pm i\omega \sigma_2} w(\xi \pm \sigma_1) + e^{\mp i\omega \sigma_1} w(\xi \pm \sigma_2) - 4w(\xi) + g'(\Phi(\xi))w(\xi).
\]

In special case $(\sigma_1, \sigma_2) = (1, 1)$ we get

\[
\mathcal{L}_\omega w(\xi) = -cw'(\xi) + (2 \cos \omega) w(\xi \pm 1) - 4w(\xi) + g'(\Phi(\xi))w(\xi)
\]

This gives $[\partial_\omega \lambda_\omega]_{\omega=0} = 0$ and $[\partial_\omega \phi_\omega]_{\omega=0} = 0$.

Eigenfunctions $\phi_\omega$ now dependent on $\omega$. But everything is quadratic in $\omega$. 
Stability - Linear System

Recall decoupled LDE

\[
\frac{d}{dt} \hat{v}_n(\omega, t) = [\hat{\Delta}^\times(\omega) \hat{v}(\omega, t)]_n + g'(\Phi(n + ct)) \hat{v}_n(\omega, t).
\]

For the Green’s function we get

\[
[G_\omega(t, t_0)]_{nn_0} = e^{\lambda_\omega(t-t_0)} \phi_\omega(n + ct) \psi^*_\omega(n_0 + ct_0) + O(e^{-\nu(t-t_0)} e^{-\nu|n + ct - n_0 - ct_0|}).
\]

Note: temporal decay of order \(O(e^{-\kappa \omega^2 \Delta t})\) since \(\text{Re} \lambda_\omega \leq -\kappa \omega^2\).

In particular, expect heat-kernel type decay in transverse direction.
Stability - Linear System

Return to full 2d linear system

\[ \dot{v}_{nl}(t) = [\Delta^x v(t)]_{nl} + g'(\Phi(n + ct))v_{nl}(t). \]

Look at initial condition

\[ v_{nl}(0) = v_{nl}^0 = (v_n^0)_l \]

with \( v^0 \in \ell^\infty(\mathbb{Z}; \ell^1(\mathbb{Z}; \mathbb{R})) \).

Norm on \( v^0 \): \( \ell^\infty \) in direction parallel to wave and \( \ell^1 \) in direction transverse to wave.

We get for \( \ell^2 \) norm in transverse direction:

\[ \|v(t)\|_{\ell^\infty(Z; \ell^2(Z; \mathbb{R}))} \sim (1 + t)^{-1/4} \|v^0\|_{\ell^\infty(Z; \ell^1(Z; \mathbb{R}))}. \]

For \( \ell^\infty \) norm in transverse direction get extra decay:

\[ \|v(t)\|_{\ell^\infty(Z; \ell^\infty(Z; \mathbb{R}))} \sim (1 + t)^{-1/2} \|v^0\|_{\ell^\infty(Z; \ell^1(Z; \mathbb{R}))}. \]
Stability - Naive Ansatz

Substituting naive perturbation Ansatz

\[ u_{nl}(t) = \Phi(n + ct) + v_{nl}(t) \]

led to

\[
\dot{v}_{nl}(t) = [\Delta^x v(t)]_{nl} + g'(\Phi(n + ct)) v_{nl}(t) \\
+ O( |v_{nl}(t)|^2 ).
\]

Linear decay of \( t^{-1/4} \) much too weak to close nonlinear argument.

However, we understand precisely the terms in Green’s function leading to slow decay:

\[
[G_\omega(t, t_0)]_{nn_0} \sim e^{\lambda_\omega(t-t_0)} \phi_\omega(n + ct) \psi^*_\omega(n_0 + ct_0).
\]

Since \( \phi_0 = \Phi' \), deformations in wave profile are the main culprit of slow decay.
Stability - Refined Ansatz

Refined perturbation Ansatz

\[ u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t). \]

Here \( \theta_l(t) \) measures deformation of wave profile (expect slow decay).

Remainder included in \( v(t) \) (expect faster decay).
Refined perturbation Ansatz

\[ u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t). \]

Normalization conditions:

\[ \sum_{n \in \mathbb{Z}} \Psi(n + ct)v_{nl}(t) = 0, \quad \text{for all } l \in \mathbb{Z}. \]

Let us shorten this to:

\[ Q_{ct}v(t) = 0 \in \ell^\infty(\mathbb{Z}; \mathbb{R}). \]

Reminder: we had \( \mathcal{L}_0 \Phi' = 0 \) and \( \mathcal{L}_0^* \Psi = 0 \) with

\[ \sum_{n \in \mathbb{Z}} \Psi(n + ct)\Phi'(n + ct) = 1. \]
Stability - Refined Ansatz

Need notation:

$$\theta^\diamond = (\theta_{l+\sigma_2} - \theta_l, \theta_{l-\sigma_2} - \theta_l, \theta_{l-\sigma_1} - \theta_l, \theta_{l+\sigma_1} - \theta_l).$$

This expression contains only differences in $\theta$. Fourier symbol for difference:

$$e^{\pm i\omega \sigma_i} - 1 = O(\omega).$$

Linear evolution for $\theta$ can be written as:

$$\dot{\theta}_l(t) = Q_{ct} L(ct + \theta) v(t) + Q_{ct} M(ct + \theta) \theta^\diamond(t) + cQ'_{ct} v(t)$$

Here we have [Very similar to naive linearization]:

$$[L(ct + \theta)v]_{nl} = [\Delta^\times v]_{nl} + g'(\Phi(n + ct + \theta_l))v_{nl}.$$  

New term [Measures effect of profile mismatches]:

$$[M(ct + \theta)\theta^\diamond]_{nl} = \Phi'(n + ct + \theta_l \pm \sigma_1)[\theta_l \pm \sigma_2 - \theta_l]$$

$$+ \Phi'(n + ct + \theta_l \pm \sigma_2)[\theta_l \mp \sigma_1 - \theta_l].$$
Stability - Refined Ansatz

Recall linear evolution for $\theta$

\[
\dot{\theta}(t) = Q_{ct}L(ct + \theta)v(t) + Q_{ct}M(ct + \theta)\theta^\diamond(t) + cQ'_{ct}v(t)
\]

with mismatch term

\[
[M(ct + \theta)\theta^\diamond]_{nl} = \Phi'(n + ct + \theta_l \pm \sigma_1)[\theta_l \pm \sigma_2 - \theta_l]
+ \Phi'(n + ct + \theta_l \pm \sigma_2)[\theta_l \mp \sigma_1 - \theta_l].
\]

Special case $(\sigma_1, \sigma_2) = (1, 0)$:

\[
[M(ct + \theta)\theta^\diamond]_{nl} = \Phi'(n + ct + \theta_l)[\theta_{l+1} + \theta_{l-1} - 2\theta_l]
= \widetilde{M}(ct + \theta)\theta^{\diamond\diamond} \quad \text{[special case]}
\]

with second-difference operator

\[
\theta^{\diamond\diamond}_l = (\theta_{l+1} + \theta_{l-1} - 2\theta_l).
\]

Similar reduction to second differences also possible for $(\sigma_1, \sigma_2) = (1, 1)$. 


Stability - Refined Ansatz

Recall linear evolution for $\theta$:

$$\dot{\theta}_l(t) = Q_{ct}L(ct + \theta)v(t) + Q_{ct}M(ct + \theta)\theta^\diamond(t) + cQ'_{ct}v(t).$$

Write $L_{ct} = L(ct + 0)$ and $M_{ct} = M(ct + 0)$. Now obtain

$$\dot{\theta}_l(t) = Q_{ct}L_{ct}v(t) + Q_{ct}M_{ct}\theta^\diamond(t) + cQ'_{ct}v(t) + h.o.t.$$

**Worst** higher order terms given by $\theta v$ and $\theta \theta^\diamond$.

In special directions $(1, 0)$ and $(1, 1)$, worst higher order terms given by $\theta v$, $\theta \theta^\diamond\diamond$ and $(\theta^\diamond)^2$. **No** $\theta \theta^\diamond$ term.
Stability - Refined Ansatz

Full linear system for \( v \) and \( \theta \):

\[
\dot{v}(t) = [I - P_{ct}]L_{ct}v(t) + [I - P_{ct}]M_{ct}\theta - cP'_{ct}v(t),
\]
\[
\dot{\theta}(t) = Q_{ct}L_{ct}v(t) + Q_{ct}M_{ct}\theta + cQ'_{ct}v(t),
\]

with \( P_{ct} = \Phi'(\cdot + ct)Q_{ct} \). Note \( P_{ct}^2 = P_{ct} \).

Write \( G(t, t_0) \) for Green’s function. Also write \( \overline{G}(t, t_0) \) for Green’s function for:

\[
\dot{w}_{nl}(t) = [L_{ct}w(t)]_{nl} = [\Delta^x w(t)]_{nl} + g'(\Phi(n + ct))w_{nl}(t)
\]

[We have already studied this system].

We then have:

\[
G(t, t_0) = \begin{pmatrix}
[I - P_{ct}]G(t, t_0)[I - P_{ct0}] & [I - P_{ct}]G(t, t_0)\Phi'(\cdot + ct_0) \\
Q_{ct}G(t, t_0)[I - P_{ct0}] & Q_{ct}G(t, t_0)\Phi'(\cdot + ct_0)
\end{pmatrix}.
\]
Stability - Refined Ansatz

Recall Green’s function:

\[
G(t, t_0) = \begin{pmatrix}
[I - P_{ct}]G(t, t_0)[I - P_{ct_0}] & [I - P_{ct}]G(t, t_0)\Phi'(. + ct_0) \\
Q_{ct}G(t, t_0)[I - P_{ct_0}] & Q_{ct}G(t, t_0)\Phi'(. + ct_0)
\end{pmatrix}.
\]

We know the slow parts of \(G(t, t_0)\). In Fourier space these are given by

\[
[G_\omega(t, t_0)]_{nn_0} \sim e^{\lambda_\omega(t-t_0)}\phi_\omega(n + ct)\psi_\omega^*(n_0 + ct_0).
\]

Now, \([I - P_{ct}]\) projects away \(\phi_0(n + ct)\). In addition, \(\psi_0(n_0 + ct_0)\) can be seen as \(Q_{ct_0}\), and we have \(Q_{ct_0}[I - P_{ct_0}] = 0\).

Roughly speaking, in Fourier space:

\[
G_\omega(t, t_0) = \begin{pmatrix}
\omega^2e^{-\kappa\omega^2(t-t_0)} & \omega e^{-\kappa\omega^2(t-t_0)} \\
\omega e^{-\kappa\omega^2(t-t_0)} & e^{-\kappa\omega^2(t-t_0)}
\end{pmatrix}.
\]
Stability - Refined Ansatz

In special direction and \((1, 1)\) we have better expansion:

\[
G_\omega(t, t_0) = \begin{pmatrix}
\omega^4 e^{-\kappa \omega^2 (t-t_0)} & \omega^2 e^{-\kappa \omega^2 (t-t_0)} \\
\omega^2 e^{-\kappa \omega^2 (t-t_0)} & e^{-\kappa \omega^2 (t-t_0)}
\end{pmatrix}.
\]

Each \(\omega\) gives \(t^{-1/2}\) extra decay. We hence expect, for initial condition \((\nu^0, \theta^0)\) that are \(\ell^1\) in transverse direction:

\[
\|\theta(t)\|_{\ell^2(Z; \mathbb{R})} \sim (1 + t)^{-1/4} \\
\|\theta^{\diamond}(t)\|_{\ell^2(Z; \mathbb{R})} \sim (1 + t)^{-3/4} \\
\|\theta^{\diamond\diamond}(t)\|_{\ell^2(Z; \mathbb{R})} \sim (1 + t)^{-5/4} \\
\|\nu(t)\|_{\ell^\infty(Z; \ell^2(Z; \mathbb{R}))} \sim (1 + t)^{-5/4},
\]

Since \textbf{worst} nonlinear terms are \(\theta \nu\), \(\theta \theta^{\diamond\diamond}\) and \((\theta^{\diamond})^2\), which all decay in \(\ell^1\) as \((1 + t)^{-3/2}\), a nonlinear argument closes easily.

Situation for \((1, 0)\) is even better, since \(\phi_\omega = \Phi'\) for all \(\omega\).
Recall rough expansion

\[ G_\omega(t, t_0) = \begin{pmatrix} \omega^2 e^{-\kappa \omega^2 t} & \omega e^{-\kappa \omega^2 t} \\ \omega e^{-\kappa \omega^2 t} & e^{-\kappa \omega^2 t} \end{pmatrix}. \]

Each \( \omega \) gives \( t^{-1/2} \) extra decay. We hence expect, for initial condition \((v^0, \theta^0)\) that are \( \ell^1 \) in transverse direction:

\[
\|\theta(t)\|_{\ell^2(Z;\mathbb{R})} \sim (1 + t)^{-1/4} \\
\|\theta^\circ(t)\|_{\ell^2(Z;\mathbb{R})} \sim (1 + t)^{-3/4} \\
\|v(t)\|_{\ell^\infty(Z;\ell^2(Z;\mathbb{R}))} \sim (1 + t)^{-3/4},
\]

**Worst** nonlinear terms now \( v\theta \) and \( \theta\theta^\circ \). Both are \( O(t^{-1}) \) in \( \ell^1 \)-transverse.

Need delicate non-linear argument.
Stability - Refined Ansatz

Need to deal with $\theta \theta^\diamond$ and $v \theta$ terms.

Key trick:

$$\theta_l(\theta_{l+1} - \theta_l) = \frac{1}{2}(\theta_{l+1}^2 - \theta_l^2 - (\theta_{l+1} - \theta_l)^2).$$

This is discrete version of

$$uu_x = \frac{1}{2}(u^2)_x,$$

heavily exploited in study of conservation laws.

Key point: $(\theta_{l+1} - \theta_l)^2$ decays very fast ($t^{-3/2}$). Difference $\theta_{l+1}^2 - \theta_l^2$ decays very slow ($t^{-1/2}$), but gives an extra $\omega$ in Fourier space which leads to more decay on Green’s function ($t^{-3/4}$ instead of $t^{-1/4}$).

$$\int_0^t (1 + t - t_0)^{-1/4}(1 + t_0)^{-1} dt_0 \sim \ln(1 + t)(1 + t)^{-1/4} \quad \text{BAD}$$

$$\int_0^t (1 + t - t_0)^{-3/4}(1 + t_0)^{-1/2} dt_0 \sim (1 + t)^{-1/4} \quad \text{GOOD}.$$
**Stability - Refined Ansatz**

Final term to deal with: $\theta v$.

Key trick: isolate slowest decaying part of $v$ from Taylor expansion of Fourier symbol. Taylor expansion not in $\omega$ but in $e^{i\omega} - 1$ in order to exploit difference structure!

Slowest decaying part of $v$ directly proportional to slowest decaying part of $\theta^\Diamond$. Can decompose:

$$v_{nl}(t) = w_{nl}(t) - i[I - P_{ct}][\partial_\omega \phi(\cdot + ct)]_{\omega=0}(\theta_{l+1}(t) - \theta_l(t)).$$

New variable $w(t)$ decays **faster** than $v$, at rate $t^{-5/4}$.

Slow part of $v(t)$ proportional to $\theta^\Diamond$. Can treat in same way as $O(\theta \theta^\Diamond)$ term!

Notice that in special directions $(1, 0)$ and $(1, 1)$, we have $v(t) = w(t)$. 
Stability in 2d

Recall Ansatz

\[ u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t). \]

**Thm.** [H., Hoffman, Van Vleck, 2012] Travelling wave \((c \neq 0)\) in any rational direction is nonlinearly stable under small perturbations

\[
\sum_{l \in \mathbb{Z}} |\theta_l(0)| \ll 1
\]
\[
\sup_{n \in \mathbb{Z}} \left[ \sum_{l \in \mathbb{Z}} |v_{nl}(0)| \right] \ll 1.
\]

Note: perturbations need to be summable in transverse direction.

We have \(\theta_l(t) \to 0\) and \(v_{nl}(t) \to 0\) as \(t \to \infty\).

In other words, deformations of interface diffuse in transverse direction.

It does NOT lead to a shift in the wave.
Stability in 2d

Recall Ansatz

\[ u_{nl}(t) = \Phi(n + ct + \theta_l(t)) + v_{nl}(t). \]

Algebraic decay rates depend on direction of propagation!

Horizontal waves \((\theta = 0)\):

\[ \theta_l(t) \sim t^{-1/2}, \quad v_{nl}(t) \sim t^{-7/4}. \]

Diagonal waves \((\theta = \pi/4)\):

\[ \theta_l(t) \sim t^{-1/2}, \quad v_{ij}(t) \sim t^{-3/2}. \]

Other rational directions: \(\text{very slow decay - delicate nonlinear analysis needed}\)

\[ \theta_l(t) \sim t^{-1/2}, \quad v_{ij}(t) \sim t^{-1}. \]
Summary

• Obtained stability in 2d for rational directions

• Only spectral conditions imposed on wave.

• Works even in absence of comparison principles.

Outlook:

• What about irrational directions?

• What about standing waves \((c = 0)\)?