Existence and Stability of Fast Pulses for the Discrete FitzHugh-Nagumo System

Hermen Jan Hupkes
University of Missouri - Columbia, MO
(Joint work with B. Sandstede)
Nerve fibres carry signals over large distances (meter range).

- Fiber has myelin coating with periodic gaps called *nodes of Ranvier*.
- Fast propagation in coated regions, but signal loses strength rapidly (mm-range).
- Slow propagation in gaps, but signal chemically reinforced.
Signal Propagation: The Model

One is interested in the potential \( U_j \) at the node sites.

Signals appear to "hop" from one node to the next [Lillie, 1925].

Ionic current has sodium and potassium component.

Electro-chemical analysis leads to the two component LDE [Keener and Sneyd, 1998]

\[
\begin{align*}
\dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a) - W_j(t), \\
\dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)],
\end{align*}
\]

posed on a 1-dimension lattice, i.e. \( j \in \mathbb{Z} \).

Potassium recovery encoded in second equation. Slow recovery → small \( \epsilon > 0 \).
Signal Propagation: Nonlinearity

Recall the dynamics:

\[
\begin{align*}
\dot{U}_j(t) &= U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a) - W_j(t), \\
\dot{W}_j(t) &= \epsilon[U_j(t) - \gamma W_j(t)].
\end{align*}
\]

Bistable nonlinearity \( g \) given by

\[ g(u; a) = u(a - u)(u - 1). \]

Parameter \( \gamma > 0 \) small so

\[ w \neq g(\gamma w; a) \]

for \( w \neq 0. \)
Recall dynamics:

\[
\begin{align*}
\dot{U}_j(t) & = U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a) - W_j(t), \\
\dot{W}_j(t) & = \epsilon[U_j(t) - \gamma W_j(t)].
\end{align*}
\]

Travelling wave Ansatz \((U_j, W_j)(t) = (u, w)(j + ct)\) leads to

\[
\begin{align*}
cu'(&\xi) = u(\xi + 1) + u(\xi - 1) - 2u(\xi) + g(u(\xi); a) - w(\xi), \\
cw'(&\xi) = \epsilon[u(\xi) - \gamma w(\xi)].
\end{align*}
\]

This is a singularly perturbed functional differential equation of mixed type (MFDE).

Interested in \textit{pulses}: \(\lim_{\xi \to \pm \infty} (u, w)(\xi) = (0, 0)\).

Previous work by [Tonnelier], [Elmer and Van Vleck]; [Carpio et al]; lot of insight; rigorous results for special cases.
Signal Propagation: FitzHugh-Nagumo LDE

**Reduction 1**: Choose $\epsilon = 0$, which gives:

\[
\begin{align*}
    cu'(\xi) &= u(\xi + 1) + u(\xi - 1) - 2u(\xi) + g(u(\xi); a) - w(\xi), \\
    cw'(\xi) &= 0,
\end{align*}
\]

admitting an equilibria-manifold $\mathcal{M} = (u, g(u; a))$.

**Fast dynamics**: $u$ varies; $w$ fixed.

**Slow dynamics**: $u$ slaved to $w$ by $g(u; a) = w$; movement only along $\mathcal{M}$.

Choose $\mathcal{M}_L$ and $\mathcal{M}_R$ as:
Signal Propagation: FitzHugh-Nagumo LDE

**Reduction 2:** Choose $\epsilon = 0$ and $W = 0$, which gives Nagumo LDE

$$\dot{U}_j(t) = U_{j+1}(t) + U_{j-1}(t) - 2U_j(t) + g(U_j(t); a).$$

Want: travelling fronts $U_j(t) = q_f(j + ct)$, which must solve MFDE

$$cq_f'(\xi) = q_f(\xi + 1) + q_f(\xi - 1) - 2q_f(\xi) + g(q_f(\xi); a),$$

$$\lim_{\xi \to -\infty} q_f(\xi) = 0, \quad \lim_{\xi \to \infty} q_f(\xi) = 1.$$

Compare to Nagumo PDE

$$\partial_t u = \partial_{xx} u + g(u, a),$$

with traveling front ODE:

$$cq_f'(\xi) = q_f''(\xi) + g(q_f(\xi); a),$$

$$\lim_{\xi \to -\infty} q_f(\xi) = 0, \quad \lim_{\xi \to \infty} q_f(\xi) = 1.$$
Signal Propagation: Comparison

**PDE**

\[ \partial_t u = \partial_{xx} u + g(u, a) \]

Travelling front \( u = q_f(x + ct) \) satisfies:

\[ cq'_f(\xi) = q''_f(\xi) + g(q_f(\xi); a) \]

Travelling fronts connecting 0 to 1:

\[ c = \frac{1}{2} \sqrt{2} \]

\[ a = \frac{1}{2} \]

**LDE**

\[ \dot{U}_j = U_{j+1} + U_{j-1} - 2U_j + g(U_j; a) \]

Travelling front \( U_j = q_f(j + ct) \) satisfies:

\[ cq'_f(\xi) = q'_f(\xi + 1) + q'_f(\xi - 1) - 2q_f(\xi) + g(q_f(\xi); a) \]

Travelling waves connecting 0 to 1:
Discrete FitzHugh-Nagumo LDE - Propagation failure

Travelling fronts for the discrete Nagumo equation connecting $0 \rightarrow 1$. 

Zero speed
Fix $0 < a < a_* $; there exists wave speed $c_* > 0$ and front $q_f$:

We now need to go back from $\mathcal{M}_R$ to $\mathcal{M}_L$.

Cubic is symmetric around inflection point $\rightarrow$ mirror $q_f$ to find back $q_b$. 
Connecting the pieces we find a singular homoclinic orbit $\Gamma_0$. 

 Signal Propagation: FitzHugh-Nagumo LDE
Main Result [H., Sandstede]: Choose $0 < a < a_\ast$ to ensure that the discrete Nagumo equation supports a front with $c > 0$. For sufficiently small $\epsilon > 0$, there is a [locally unique] stable travelling pulse solution $\Gamma(\epsilon)$ to the discrete FitzHugh–Nagumo LDE that bifurcates off $\Gamma_0$ and winds around $\Gamma_0$ once.
Result generalizes classic existence + stability theorem for FitzHugh-Nagumo PDE [Carpenter], [Hastings], [Yanagida] ('70s and '80s)

\[
\begin{align*}
U_t &= U_{xx} + g(U; a) - W, \\
W_t &= \epsilon[U - \gamma W].
\end{align*}
\]

'Modern' existence proof [Jones et al] uses Exchange Lemma to show transverse intersection of manifolds \( \mathcal{W}^u(0) \) and \( \mathcal{W}^s(\mathcal{M}_L) \).
The program

Main goal: lift geometric singular perturbation theory to MFDEs.

• Ill-posedness: care must be taken to define unstable / stable manifolds.

• Track intersections of $\infty$-dim stable / unstable manifolds.

• Exchange Lemma: Fenichel coordinates unavailable in infinite dimensions.

• Evans function: Not available for MFDEs.

Main ingredients:

• Suitable finite dimensional subspaces of $C([-1, 1], \mathbb{R})$.

• Analytical underpinning for geometrical constructions.

• Direct construction of potential eigenfunctions.
Existence: Step 1 - Breaking the front

Varying $\epsilon$ and $c$ breaks orbit $q_f$ into quasi-front solution: two parts $(u^-, w)$ and $(u^+, w)$.

Want to contain jump in some finite-dimensional $\Gamma_f \subset C([-1, 1], \mathbb{R})$. 

$\Gamma_f$ is depicted in the diagram.
Existence: Step 1 - Breaking the front

Construction based upon exponential dichotomies on $\mathbb{R}$ for linearization

$$cv' (\xi) = v(\xi + 1) + v(\xi - 1) - 2v(\xi) + g'(q_f (\xi)) v(\xi).$$

Thm. [Mallet-Paret and Verduyn Lunel, 2001]:

$$C([-1, 1], \mathbb{R}) = \widehat{P} \leftarrow \oplus \widehat{Q} \rightarrow \oplus \{q'_f\} \oplus \Gamma_f.$$
Existence: Step 2 - Breaking the back

Similarly, can construct quasi-back solutions.

Extra degree of freedom $w_0 \approx w_*$, lifts back up and down.
Existence: Step 3 - Exchange Lemma

Half-way along $\mathcal{M}_R$, quasi-front and quasi-back miss each other by $O(e^{-1/\epsilon})$. Slight perturbation yields quasi-solutions:
Existence: Step 3 - Exchange Lemma

Construction uses seven distinct intervals.

$\xi = 3\xi_* + T$

$\xi = 2\xi_* + T$

$\xi = \xi_* + T$

$u_b = u_{qb}^+ + v_b^-\quad Q_{b,\rightarrow}(\xi_*)\quad P_{b,\leftarrow} R_{b,\rightarrow}(-\xi_*)$

$u_{xc} = u_{q_b}^- + v_b^0\quad u_{xc} = u_{q_b}^- + v_b^0$

$P_{f,\leftarrow} \Gamma_f$

$Q_{f,\rightarrow}(\xi_*)$ \hspace{1cm} $S_{f,\leftarrow}(\xi_*)$

$u_f = u_{q_f}^- + v_f^-\quad u_{q_f}^+ + v_f^0$

$P_{R,\leftarrow}^{fb}(u_{hw})\quad Q_{R,\rightarrow}^{fb}(0)$

matching q-front and q-back
Existence: Step 4 - Bifurcation equations

The jumps in $\Gamma_f$ and $\Gamma_b$ can be split into two parts:

- Construction of quasi-fronts and quasi-backs
  - Contribution of $O(\epsilon + |c - c_*| + |w_0 - w_*|)$.

- Modification due to Exchange Lemma
  - Contribution + derivatives are $O(e^{-1/\epsilon})$.

System to solve is hence to leading order

\[
\begin{align*}
M_1(c - c_*) &= M_2\epsilon \\
M_3(c - c_*) &= M_4(w_0 - w_*) + M_5\epsilon
\end{align*}
\]

The sign of $M_1 - M_5$ can be read off from Melnikov integrals.

Three unknowns; two equations $\rightarrow$ curve of solutions $(\epsilon, c(\epsilon))$. 
Stability

We have hence constructed travelling wave solutions

\[(U_j, W_j)(t) = (\bar{u}(\epsilon), \bar{w}(\epsilon))(j + c(\epsilon)t).\]

Waves are shift-periodic with respect to the lattice

\[(U_j, W_j)\left(t + 1/c(\epsilon)\right) = (U_{j+1}, W_{j+1})(t).\]

Possible to use shift-periodic Floquet theory to study stability [Chow, Mallet-Paret, Shen].

However, we 'pretend' that \(j \in \mathbb{Z}\) is continuous and study the eigenvalue MFDE

\[
c(\epsilon)u'(\xi) = u(\xi - 1) + u(\xi + 1) - 2u(\xi) + g'(\bar{u}(\epsilon)(\xi))u(\xi) - w(\xi) - \lambda u(\xi),
\]

\[
c(\epsilon)w'(\xi) = \epsilon(u(\xi) - \gamma w(\xi)) - \lambda w(\xi),
\]

in comoving frame \(\xi = j + ct\). Write as

\[\mathcal{L}(\epsilon)(u, w) = \lambda(u, w).\]
Stability - Relation between points of view

Rewrite LDE as \( (\dot{U}, \dot{W})(t) = \mathcal{F}\left(U(t), W(t)\right) \) posed on \( \ell^\infty \).

**Green’s function** \( G_{jj_0}(t, t_0, \epsilon) \) is unique solution to linearized LDE

\[
(\dot{U}, \dot{W})(t) = D\mathcal{F}\left((\bar{u}(\epsilon), \bar{w}(\epsilon))(\cdot + c(\epsilon)t)\right)\left(U, W\right)(t) \\
(U_j, W_j)(t_0) = \delta_{jj_0}.
\]

**Resolvent kernel** \( G_\lambda(\xi, \xi_0, \epsilon) \) is unique solution to linearized MFDE

\[
(\mathcal{L}(\epsilon) - \lambda)G_\lambda(\cdot, \xi_0, \epsilon) = \delta(\xi - \xi_0).
\]

Lattice does not see modulations \( e^{2\pi i \xi} \). In particular,

\[
G_{\lambda+2\pi i c(\epsilon)}(\xi, \xi_0, \epsilon) = e^{2\pi i (\xi_0 - \xi)} G_\lambda(\xi, \xi_0, \epsilon).
\]

**Thm.** [Benzoni-Gavage, Huot, Rousset] For \( \gamma \gg 1 \) and \( t > 0 \),

\[
G_{jj_0}(t, t_0, \epsilon) = \frac{-1}{2\pi i} \int_{\gamma - i\pi c(\epsilon)}^{\gamma + i\pi c(\epsilon)} e^{\lambda(t-t_0)} G_\lambda(j + ct, j_0 + ct_0, \epsilon) d\lambda.
\]
Stability

Goal is to shift contour of integration in

\[ G_{jj_0}(t, t_0, \epsilon) = \frac{-1}{2\pi i} \int_{\gamma-i\pi c(\epsilon)}^{\gamma+i\pi c(\epsilon)} e^{\lambda(t-t_0)} G_{\lambda}(j + ct, j_0 + ct_0, \epsilon) d\lambda \]

to the line \( \gamma = -\delta_0 \). Need to extend resolvent kernel \( G_{\lambda} \) meromorphically through imaginary axis.

Will show: Spectrum of \( \mathcal{L}(\epsilon) \) admits gap.

Translational eigenvalues at \( 2\pi ic(\epsilon)\mathbb{Z} \) contribute simple poles to resolvent kernel \( G_{\lambda}(\xi, \xi_0, \epsilon) \).
Stability

Goal is to characterize eigenvalues for $\mathcal{L}(\epsilon)$ in three regions $R_1$, $R_2$ and $R_3$ simultaneously for all small $\epsilon > 0$ by direct construction.

Essential spectrum is $O(\epsilon)$ to left of imaginary axis.

Push out of the way by using exponential weights, i.e., choose small $\eta > 0$ and look for solutions $\Lambda(\epsilon)(u, w) = \lambda(u, w)$ that behave as $(u, w)(\xi) = O(e^{\eta \xi})$ as $\xi \to \pm \infty$. 
Stability - Resonance pole or eigenvalue

Translational eigenvalue at $\lambda = 0$.

The pulse $(\bar{u}, \bar{w})(\epsilon)$ can be thought of as bound state of front $q_f$ and back $q_b$.

Expect second potential eigenvalue $\lambda_2 = O(\epsilon)$, with eigenfunction centered on the back $q_b$.

Whether $\lambda_2$ is an eigenvalue or resonance pole depends on location with respect to imaginary axis.

Our direct construction of eigenfunctions yields explicit expression for the speeds with which $\lambda_2$ and the essential spectrum move to the left.
Stability - Resonance pole or eigenvalue

All three scenario’s can occur.
**Situation (i):** $\lambda_2$ is an eigenvalue to the right of essential spectrum. Perturbations that change only the position of the back will decay without interacting with the front.

Other perturbations lead to a translation of the pulse profile and a movement of the back relative to the front.

**Situation (ii):** $\lambda_2$ is eigenvalue. Effect should still be felt for localized perturbations, affects relative position of front and back. Essential spectrum transports perturbations of background state $(u, w) = 0$ to $j = \infty$.

**Situation (iii):** $\lambda_2$ is resonance pole. Unclear. More detailed analysis of resolvent kernel may lead to insight.
Summary / Outlook

- Travelling pulses for discrete FHN constructed using $\infty$-d Exchange Lemma.
- Stability established by direct construction of potential eigenfunctions.

Number of issues to explore:

- Multi-pulses, homoclinic blow-up etc in other singularly perturbed lattice problems.
- What happens to pulses as propagation failure region is encountered?
FitzHugh-Nagumo PDE: Slow Pulses

Recall the travelling wave ODE

\[
\begin{align*}
    u' &= v, \\
    v' &= cv - g(u; a) + w, \\
    w' &= \frac{\epsilon}{c}(u - \gamma w).
\end{align*}
\]

In the singular limit \( c \to 0 \) and \( \frac{\epsilon}{c} \to 0 \), one finds an additional slow-singular orbit \( \Gamma_{0}^{sl} \).
Conjecture [Yanagida]: fast and slow branches are connected.