Introduction to Algebraic Topology, Fall 2018, practice exercises - week 11

Exercise 1. Let Y be a topological graph, i.e. a compact metric space such that for each $p \in Y$ there exist a positive integer n and $\epsilon \in \mathbb{R}_{>0}$ such that p possesses an open neighborhood $U \subseteq Y$ together with an isometry $U \xrightarrow{\sim} S(n,\epsilon)$, where $S(n,\epsilon)$ is the star-shaped set

$$S(n,\epsilon) = \{z \in \mathbb{C} \, : \, \text{there exist } 0 \leq t < \epsilon \text{ and } k \in \mathbb{Z} \text{ such that } z = te^{2\pi i k/n}\} \, ,$$

endowed with the path metric. For each $p \in Y$ the integer n is uniquely determined, and is called the *valence* of p, notation $v(p)$. Let $V \subset Y$ be the set of points $p \in Y$ with $v(p) \neq 2$. We call V the set of *vertices* of Y.

(i) Show that V is a finite (possibly empty) set.

The space $Y \setminus V$ has a finite number of connected components, each isometric with an open interval. The closure in Y of a connected component of $Y \setminus V$ is called an *edge* of Y. Let $n \in \mathbb{Z}_{>0}$. If Y has the property that for all $p \in V$ we have $v(p) = n$ we say that Y is n-regular. Assume that Y is 4-regular. Suppose furthermore that the edges of Y have been assigned labels a and b and orientations in such a way that the local picture near each vertex is the same as in the figure-eight (see Fulton, bottom of p. 190), so there is an a-edge oriented toward the vertex, an a-edge oriented away from the vertex, a b-edge oriented toward the vertex, and a b-edge oriented away from the vertex. In this case we see that Y is 2-oriented. In the following, when we say *graph* we mean *connected topological graph*.

(ii) Draw some 4-regular 2-oriented graphs with $m = 1, 2, 3, \ldots$ vertices.

(iii) Show that every 4-regular graph can be 2-oriented.

Let X be the figure-eight.

(iv) Let $p \colon Y \to X$ be a covering with Y connected. Show that Y is a 4-regular (and 2-oriented) graph.

(v) Show that for every 4-regular (and 2-oriented) graph Y, there exists a covering map $p \colon Y \to X$.

(vi) Construct coverings $p \colon Y \to X$ of degree $m = 1, 2, 3, \ldots$.

Exercise 2. (cf. Problem 1 in Looijenga's syllabus) Let X be a path-connected topological space. Show that the map $\deg \colon C_0(X) \to \mathbb{Z}$ given by sending a 0-chain $\sum_{x \in X} a_x \cdot x$ to $\sum_{x \in X} a_x$ induces an isomorphism $H_0(X) \xrightarrow{\sim} \mathbb{Z}$.

Exercise 3. Do Problems 2, 3, 5, 7 and 10 from Looijenga's syllabus.

Exercise 4. Let X be a connected topological graph (see Exercise 1). Show that $H_1(X)$ is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{Z}_{>0}$. Hint: use the main point of Problem 5 in Looijenga's syllabus: for every path-connected topological space X, and every base-point $x \in X$, the homology group $H_1(X)$ can be (canonically) identified with the *abelianization* of $\pi_1(X, x)$.

Exercise 5. Draw a picture of the first barycentric subdivision of the standard 3-simplex Δ^3, and count the number of simplices in it.