Introduction to Algebraic Topology, Fall 2018, practice exercises - week 8

Let X be a topological space which is **connected** and **locally path connected**. Let $x \in X$.

Exercise 1. This exercise is a follow-up on Exercise 2 of homework III. Let $p\colon Y \to X$ be a covering, and assume that Y is connected. Part (ii) of Exercise 2 of homework III produces a natural isomorphism of groups $\mathrm{Aut}(Y/X) \xrightarrow{\sim} \mathrm{Aut}_{\pi_1(X,x)}(Y_x)$. The purpose of this exercise is to "compute" Y_x as a $\pi_1(X,x)$-set.

Let π be a group, and let $H \subseteq \pi$ be a subgroup.

(i) Show that the set π/H of cosets $H \cdot g$, for $g \in \pi$, has a natural structure of right π-set.

(ii) Show that the right π-set π/H is transitive.

(iii) Let S be a transitive right π-set. Show that S is isomorphic, as a right π-set, with a coset space π/H, for some $H \subseteq \pi$.

(iv) Let $S = \pi/H$ be a transitive right π-set. Let $N_\pi(H) \subseteq \pi$ be the normalizer of H in π (see the syllabus Algebra 1, edition 2017, p. 65). Exhibit an isomorphism of groups

$$\phi\colon N_\pi(H)/H \xrightarrow{\sim} \mathrm{Aut}_\pi(S)\,.$$

A correct answer to this question contains the following: a proof that the map ϕ that you propose is well-defined; a proof that your ϕ is a group homomorphism; a proof that your ϕ is injective and surjective.

Now let $\pi = \pi_1(X,x)$. Fix a basepoint $y \in Y$ such that $p(y) = x$.

(v) Show that Y_x is isomorphic as a right π-set with the transitive π-set π/H, where $H = \mathrm{Stab}_y = p_*\pi_1(Y,y) \cong \pi_1(Y,y)$.

(vi) Show that there exists a natural isomorphism of groups

$$N_{\pi_1(X,x)}(\pi_1(Y,y))/\pi_1(Y,y) \xrightarrow{\sim} \mathrm{Aut}(Y/X)\,.$$

Exercise 2. Assume that X admits a universal covering space. Let $H \subseteq \pi_1(X,x)$ be a subgroup and assume that H has finite index d in $\pi_1(X,x)$. Show that the covering $Y \to X$ corresponding to H via the Galois correspondence is a d-sheeted covering.

Exercise 3. (cf. Fulton, Exercise 11.12) Let $q\colon Z \to X$ and $p\colon Y \to X$ be coverings, and let $g\colon Z \to Y$ be a morphism of coverings over X, that is, we have g is continuous, and $q = p \circ g$. Assume that Y is connected. Show that g is a covering.

Exercise 4. Let $q\colon Z \to X$ and $p\colon Y \to X$ be coverings, and let $g\colon Z \to Y$ be a morphism of coverings over X. Show that g is open. When both p, q are G-coverings for the same group G, show that g is an isomorphism of coverings.

Exercise 5. (cf. Fulton, Exercise 11.24) Let $q\colon Z \to X$ and $p\colon Y \to X$ be G-coverings for the same group G. Let $g_1, g_2\colon Z \to Y$ be two morphisms of G-coverings, i.e. morphisms of coverings that commute with the G-action. Then by the previous exercise g_1, g_2 are in fact isomorphisms of G-coverings. Assume X is connected and let $z \in Z$ be such that $g_1(z) = g_2(z)$. Show that $g_1 = g_2$. In particular, if X is connected the automorphism group of a pointed G-covering of X (where automorphism means G-automorphism) is trivial.

Exercise 6. Let $p\colon Y \to X$ be a G-covering with Y connected. Let $\mathrm{Aut}_G(Y/X)$ be the group of G-automorphisms of Y/X, i.e. the group of automorphisms of Y/X that commute with the G-action. Show that $\mathrm{Aut}_G(Y/X) = Z(G)$, where $Z(G) \subset G$ is the center of G.

For $n \in \mathbb{Z}_{\geq 0}$ we put $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Both S^n and D^n are equipped with the induced topology.

Exercise 7. Let $n \in \mathbb{Z}_{\geq 0}$. Let C^{n+1} be the space obtained from $[0,1] \times S^n$ by contracting the subspace $\{0\} \times S^n$ to a point. More precisely C^{n+1} is the quotient space $([0,1] \times S^n)/\sim$ for the equivalence relation $\sim$ on $[0,1] \times S^n$ given by $y \sim y' \Leftrightarrow y = y'$ or $y, y' \in \{0\} \times S^n$.

(i) Exhibit a homeomorphism $C^{n+1} \xrightarrow{\sim} D^{n+1}$, where D^{n+1} is the closed unit ball in $\mathbb{R}^{n+1}$.

Let Σ^{n+1} be the quotient space $([0,1] \times S^n)/\approx$ for the equivalence relation $\approx$ on $[0,1] \times S^n$ given by $y \approx y' \Leftrightarrow y = y'$ or $y, y' \in \{0\} \times S^n$ or $y, y' \in \{1\} \times S^n$.

(ii) Exhibit a homeomorphism $\Sigma^{n+1} \xrightarrow{\sim} S^{n+1}$.

(iii) Show that S^{n+1} can be obtained from D^{n+1} by contracting the boundary $S^n \subset D^{n+1}$ to a point. Can you give an explicit continuous map $f\colon D^{n+1} \to S^{n+1}$ that realizes this identification?

(iv) Let $p \in S^{n+1}$ be any point. Show that $S^{n+1} \setminus \{p\}$ is homeomorphic with the open unit ball in $\mathbb{R}^{n+1}$ and hence with $\mathbb{R}^{n+1}$ itself.

Exercise 8. Show that there do not exist two open subsets $U, V \subset X = S^1$ such that the following conditions are satisfied: $X = U \cup V$; both U and V are simply connected; the intersection $U \cap V$ is path-connected.

Exercise 9. Let B, C be two groups. Show that $B * C$ and $C * B$ are canonically isomorphic.