Exam Introduction to Algebraic Topology

21 January 2019, 10:00 – 13:00

Please answer the following ten (sub)questions. Each subquestion is worth five points.
Motivate your answers. Try to be both concise and complete.
Formulate precisely any result proved in class that you use.
For $n \in \mathbb{Z}_{\geq 0}$ we set $S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\| = 1\} \subset \mathbb{R}^{n+1}$, equipped with the subspace topology.

Exercise 1. Assume that $p\colon Y \to X$ is a G-covering and $q\colon W \to V$ is an H-covering. Show that the map $Y \times W \to X \times V$ given by $(y, w) \mapsto (p(y), q(w))$ is a $G \times H$-covering.

Exercise 2. Let $Y = S^3$. We view Y as the unit sphere $\{(w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 = 1\}$ in $\mathbb{C}^2$. Let $n \in \mathbb{Z}_{\geq 1}$. Let $\zeta = \exp(2\pi i/n)$, and let $\boldsymbol{\mu}_n = \langle \zeta \rangle \subset \mathbb{C}^*$ be the group of n-th roots of unity in $\mathbb{C}^*$. The group $\boldsymbol{\mu}_n$ acts evenly on Y by putting $\zeta \cdot (w_1, w_2) = (\zeta w_1, \zeta w_2)$. (You do not need to prove this.) Write $X_n = Y/\boldsymbol{\mu}_n$ and write $p_n\colon Y \to X_n$ for the canonical projection.

(a) What is the fundamental group of X_n?

Let $f\colon X_4 \to X_5$ be a continuous map.

(b) Show that there exists a continuous map $\widetilde{f}\colon X_4 \to Y$ such that $f = p_5 \circ \widetilde{f}$.

Exercise 3. Let X be the figure-eight ∞ with base-point the point x where the two circles meet.

(a) Describe the fundamental group of X (with base-point x).

Set $G = \mathbb{Z}/2\mathbb{Z}$.

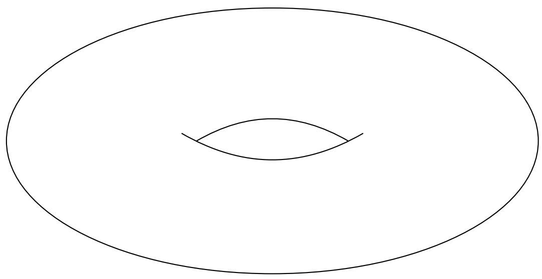
(b) Determine the number of isomorphism classes of pointed G-coverings of (X, x).

(c) Draw a representative of each isomorphism class of pointed G-coverings of (X, x).

Exercise 4.

(a) State the theorem about the Mayer-Vietoris long exact sequence.

(b) Compute the homology groups of the torus $T = S^1 \times S^1$, that is, of the surface



in $\mathbb{R}^3$.

Exercise 5. Let $f\colon S^2 \to S^2$ be a continuous map. Assume that f has no fixed points.

(a) Show that f is homotopic to the (antipodal) map $S^2 \to S^2$ given by $x \mapsto -x$.

(b) Show that there is no even action of the group $\mathbb{Z}/3\mathbb{Z}$ on S^2.