Graphs, Matrices, Determinants, and Pfaffians

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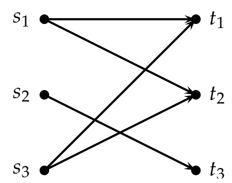
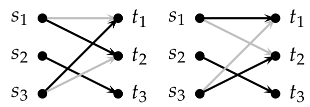
Consider an $n \times n$ matrix A with entries a_{ij} in some commutative ring R. We may associate to A a directed graph G_A in the following way: The vertices of G_A are $\{s_1, \ldots, s_n, t_1, \ldots, t_n\}$, and there is an edge $s_i \to t_j$ with weight a_{ij} whenever a_{ij} is nonzero. The determinant of A is the sum over all ways of choosing n edges $e_i : s_i \to t_{\sigma(i)}$, with the $\sigma(i)$ all distinct, of the product of the edge weights $a_{i\sigma(i)}$, with a sign in accordance with the the sign of the resulting permutation $\sigma \in \mathrm{S}_n$.

Example 1

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

The graph G_A:





$\det(A) = a_{12}a_{23}a_{31} - a_{11}a_{23}a_{32}$

In general, given two n-tuples $S = (s_1, \ldots, s_n)$ and $T = (t_1, \ldots, t_n)$ of vertices of a directed graph, a *connection* from S to T is a set K of edges that form n vertex-disjoint directed paths from a vertex in S to a vertex in T. Each connection K has an associated permutation σ_K, defined by $t_{\sigma_K(i)}$ being the endpoint of the path in K starting at s_i; define $\mathrm{sgn}(K)$ as the sign of the permutation σ_K. Then our earlier observation can be written

$$\det(A) = \sum_{K:S_A \to T_A} \mathrm{sgn}(K) \prod_{e \in K} \mathrm{wt}(e), \tag{1}$$

where we are summing over all connections K from $S_A = (s_1, \ldots, s_n)$ to $T_A = (t_1, \ldots, t_n)$ in G_A, and denoting the weight of edge e by $\mathrm{wt}(e)$.

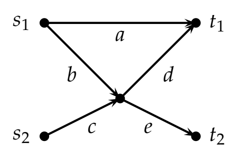
We may generalize Equation (1) as follows. For any directed acyclic graph G with two n-tuples of vertices S and T, we can produce an $n \times n$ matrix $\mathrm{M}_G(S;T)$ whose ijth entry is a sum over paths $\sum_{P:s_i \to t_j} \prod_{e \in P} \mathrm{wt}(e)$. Then

$$\det(\mathrm{M}_G(S;T)) = \sum_{K:S \to T} \mathrm{sgn}(K) \prod_{e \in K} \mathrm{wt}(e). \tag{2}$$

Equation (2) is called the *Lindström-Gessel-Viennot lemma*.

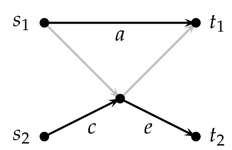
Example 2

A weighted directed acyclic graph G:



$$\mathrm{M}_G(s_1, s_2; t_1, t_2) = \begin{pmatrix} a + bd & be \\ cd & ce \end{pmatrix}$$

There is only one connection from (s_1, s_2) to (t_1, t_2), with positive sign:



$\det(\mathrm{M}_G(s_1, s_2; t_1, t_2)) = ace.$

An easy corollary is that the determinant is multiplicative: If we have two $n \times n$ square matrices A and B, and let G be the concatenation of G_A and G_B formed by identifying T_A with S_B, then the matrix $\mathrm{M}_G(S_A; T_B)$ is just the product AB and the left-hand side of Equation (2) is just $\det(AB)$. However, every connection from S_A to T_B decomposes as a connection from S_A to T_A and a connection from S_B to T_B, so it is easy to show that the right-hand side of Equation (2) is $\det(A)\det(B)$. The more complicated Cauchy-Binet formula for the determinant of AB when A and B are not necessarily square is just as easy to prove in this way.

A seemingly unrelated fact is that the determinant of any alternating matrix A is the square of a polynomial in the above-diagonal entries: this polynomial is called the *Pfaffian* $\mathrm{Pf}(A)$. The Pfaffian also has a graphical formula analogous to Equation (1), but there is no known generalization analogous to the Lindström-Gessel-Viennot lemma that would allow one to easily prove identities such as $\mathrm{Pf}(BAB^T) = \det(B)\,\mathrm{Pf}(A)$ or its variations.

Example 3

An alternating matrix A:

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

$\det(A) = (af - be + cd)^2$