Squarefree groups

This project is suitable for a student who likes groups, in particular finite groups.

A positive integer is called *squarefree* if it is not divisible by the square of any prime number. A finite group is sometimes called squarefree if its order is squarefree. A classical theorem asserts that each such group is the semidirect product of two cyclic groups. The proof of that theorem is not completely trivial, and it is part of the project.

However, there is a less naïve notion of a "squarefree" group, in which one does not just look at the order of the group but at the structure of the group. Here the notion of a prime number, which may be defined as a positive integer that has exactly two divisors, is replaced by the notion of a *simple group*, which is a group that has exactly two normal subgroups.

Every finite group G has, for some non-negative integer t, a chain of subgroups $\{1\} = G_0 \subset G_1 \subset \ldots \subset G_t = G$ with the property that for all i with $0 < i \leq t$ the group G_{i-1} is a normal subgroup of G_i and G_i/G_{i-1} is simple; such a chain is called a *composition series* of G. We now call a finite group G *squarefree* if there is such a composition series with the property that the t simple groups G_i/G_{i-1} are pairwise non-isomorphic; one can in fact show that this is independent of the choice of the composition series. If we think of G as being "built up" from the quotients G_i/G_{i-1}, then this is the same as saying that each simple group "occurs" at most once in G.

It turns out that a finite group has squarefree order if and only if it is solvable and squarefree. The main purpose of the project is to find and prove an analogue, for general squarefree finite groups, of the classical theorem about finite groups of squarefree order that we mentioned above.

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