

# **Algebraic Topology—an introduction**

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By a *space* we will always mean a topological space. Maps between spaces are supposed to be continuous unless otherwise stated.

## 1. Defining Singular Homology

**Simplices.** We recall that a subset  $A$  of  $\mathbb{R}^n$  or more generally, of a real vector space, is called *convex* if for any pair  $p, q \in A$  of its points, the segment spanned by them,  $[p, q] := \{(1-t)p + tq \mid 0 \leq t \leq 1\}$ , is also contained in  $A$ . It is then an exercise to show that whenever  $p_1, \dots, p_k$  lie in  $A$ , then so does every linear combination of the form  $t_1 p_1 + \dots + t_k p_k$  with  $t_i \geq 0$  and  $\sum_i t_i = 1$  (this is called a *convex linear combination* of  $p_1, \dots, p_k$ ). For an arbitrary subset  $A$  of a real vector space, we define its *convex hull*  $[A]$  as the collection of convex linear combinations of finite subsets of  $A$ . Another exercise consists in showing that this convex hull is, as the name suggests, convex. It is indeed the smallest convex set containing  $A$ .

The *standard  $k$ -simplex* is the subset of  $\mathbb{R}^{k+1}$  defined by

$$\Delta^k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid t_i \geq 0, i = 0, \dots, k, t_0 + \dots + t_k = 1\}.$$

Notice that this is also the convex hull of the basis vectors  $e_0, \dots, e_k \in \mathbb{R}^{k+1}$ . So  $\Delta^0$  is the singleton  $\{e_0\}$ ,  $\Delta^1$  the line segment spanned by  $e_0$  and  $e_1$ ,  $\Delta^2$  the full triangle spanned by  $e_0, e_1, e_2$  etc. As a closed and bounded subset of  $\mathbb{R}^n$ ,  $\Delta^k$  is compact. The notion of a simplex has a modest generalization as follows. Suppose given a nonempty *finite* subset  $P$  of some real vector space  $V$  that is *affine independent* in the sense that if certain scalars  $(c_p \in \mathbb{R})_{p \in P}$  with zero sum  $\sum_p c_p = 0$  yield a dependence relation  $\sum_{p \in P} c_p p = 0$ , then  $c_p = 0$  for all  $p \in P$  (for instance, if  $p_1, \dots, p_k$  are linearly independent in  $V$ , then  $0, p_1, \dots, p_k$  are affine independent). Then its convex hull

$$[P] := \left\{ \sum_{p \in P} t_p p \mid t_p \geq 0, \sum_{p \in P} t_p = 1 \right\},$$

is called the *simplex spanned by  $P$* . The *vertices* of  $[P]$  are the elements of  $P$  and the *dimension* of  $[P]$  is  $|P| - 1$ . The coefficients  $t_p$  of an element  $x = \sum_{p \in P} t_p p$  are called its *barycentric coordinates*. If  $k := |P| - 1$  and  $p_0, \dots, p_k$  is an enumeration of the elements of  $P$ , then we have a map

$$\langle p_0, \dots, p_k \rangle : \Delta^k \rightarrow [P], \quad t \in \Delta^k \mapsto t_0 p_0 + \dots + t_k p_k.$$

We claim that it is homeomorphism: it is evidently continuous and it is also bijective: surjectivity is clear and if  $t_0 p_0 + \dots + t_k p_k = t'_0 p_0 + \dots + t'_k p_k$  (with  $\sum_{p \in P} t_p = 1 = \sum_{p \in P} t'_p$ ), then  $\sum_{p \in P} (t_p - t'_p) p = 0$  with  $\sum_{p \in P} (t_p - t'_p) = 0$ , and so  $t_p - t'_p = 0$  for all  $p \in P$ ; in other words,  $(t'_0, \dots, t'_k) = (t_0, \dots, t_k)$ . The compactness of  $\Delta^k$  and the Hausdorff property of the target space imply that  $\langle p_0, \dots, p_k \rangle$  is a homeomorphism. Notice that this map depends on the enumeration of the members of  $P$ : a different numbering yields a different map. If  $Q \subset P$  is a nonempty subset of  $P$ , then clearly  $Q$  is also affine independent. The simplex it spans,  $[Q]$ , is contained in  $[P]$ ; it is called a *facet* of  $[P]$ ; if  $Q$  has just one element less than  $P$  (so that  $\dim Q = \dim P - 1$ ), then we also call  $Q$  a *face* of  $[P]$ .

**Singular chains.** We start with a definition.

DEFINITION 1.1. A *singular  $k$ -simplex* of a space  $X$  is a (by our convention continuous) map  $\sigma : \Delta^k \rightarrow X$ .

Thus for  $0 \leq i_0 < \dots < i_l \leq k$ ,  $\langle e_{i_0}, \dots, e_{i_l} \rangle$  is a singular  $l$ -simplex of  $\Delta^k$ .

The free abelian group generated by the singular  $k$ -simplices of  $X$  will be denoted  $C_k(X)$  and an element of  $C_k(X)$  is called a  *$k$ -chain* on  $X$ . So any  $k$ -chain  $c$  on  $X$  can be written  $\sum_{\sigma} c_{\sigma} \sigma$ , where  $\sigma$  runs over all singular  $k$ -simplices of  $X$  and  $c_{\sigma} \in \mathbb{Z}$ , but such that  $c_{\sigma} \neq 0$  for all but finitely many  $\sigma$ 's (thus rendering the sum finite). We call the (finite) set of singular simplices  $\sigma$  with  $c_{\sigma} \neq 0$  the *support* of  $c$ . These groups can be quite large, for instance  $C_0(X)$  is the free abelian group generated by  $X$ . We stipulate that  $C_k(X) = 0$  when  $k = -1, -2, \dots$ .

If  $f : X \rightarrow Y$  is a map between spaces, then for every singular  $k$ -simplex  $\sigma$  of  $X$ , the composite map  $f\sigma$  is a singular  $k$ -simplex of  $Y$ . By extending this additively we get a homomorphism

$$f_* : C_k(X) \rightarrow C_k(Y), \quad \sum_i c_i \sigma_i \mapsto \sum_i c_i (f\sigma_i).$$

We sometimes denote this homomorphism somewhat less ambiguously by  $C_k(f)$ . Let us make the following simple observation (which, if you are familiar with the language of category theory, can be expressed as asserting the functorial character of the formation of the  $k$ -chains).

PROPOSITION 1.2. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are maps between spaces, then  $(gf)_* = g_* f_*$ . Moreover, the identity map  $1_X : X \rightarrow X$  induces the identity map in  $C_k(X)$ .  $\square$

The 'boundary' of  $\Delta^k$  can be thought of as a union of its faces, these are  $(k-1)$ -simplices,  $k+1$  in number, gotten by setting one of the barycentric coordinates equal to zero. We use this to describe a homomorphism  $C_k(X) \rightarrow C_{k-1}(X)$ . First of all, we define a  $(k-1)$ -chain  $d\langle e_0, \dots, e_k \rangle$  on  $\Delta^k$  by

$$d\langle e_0, \dots, e_k \rangle := \sum_{i=0}^k (-1)^i \langle e_0, \dots, \widehat{e}_i, \dots, e_k \rangle,$$

where the roof ( $\frown$ ) indicates that the element it covers must be suppressed:

$$\begin{aligned} \langle e_0, \dots, \widehat{e}_i, \dots, e_k \rangle &: \Delta^{k-1} \rightarrow \Delta^k, \\ (t_0 e_0 + \dots + t_{k-1} e_{k-1}) &\mapsto (t_0 e_0 + \dots + t_{i-1} e_{i-1} + t_i e_{i+1} + \dots + t_{k-1} e_k). \end{aligned}$$

Given a singular  $k$ -simplex  $\sigma : \Delta^k \rightarrow X$ , then we define  $d(\sigma) \in C_{k-1}(X)$  by the identity

$$(1) \quad d(\sigma) = \sigma_*(d\langle e_0, \dots, e_k \rangle) = \sum_i (-1)^i \sigma \langle e_0, \dots, \widehat{e}_i, \dots, e_k \rangle.$$

Its (unique) linear extension

$$d_k : C_k(X) \rightarrow C_{k-1}(X),$$

is a homomorphism, called the *boundary homomorphism*. For  $k \leq 0$ ,  $d_k$  must be the zero map, of course.

PROPOSITION 1.3. For any map  $f : X \rightarrow Y$  we have  $df_* = f_* d$ .

PROOF. As this needs only to be checked for a set of generators, it suffices to verify this for an arbitrary singular  $k$ -simplex  $\sigma$  on  $X$ , only. We have

$$\begin{aligned} d(f_*(\sigma)) &= d(f\sigma) = (f\sigma)_*(d\langle e_0, \dots, e_k \rangle) \quad \text{by (1)} \\ &= f_*(\sigma_*(d\langle e_0, \dots, e_k \rangle)) \quad \text{by (1.2)} \\ &= f_*(d\sigma), \quad \text{by (1) again.} \end{aligned}$$

□

The signs that appear in the definition of the boundary homomorphism have been chosen in such a manner that the following holds.

PROPOSITION 1.4. *We have  $d_{k-1}d_k = 0$  for all  $k \in \mathbb{Z}$ .*

PROOF. This too needs only be verified for a single singular  $k$ -simplex  $\sigma$ . According to 1.3 we have  $dd(\sigma) = dd(\sigma_*(e_0, \dots, e_k)) = \sigma_*(dd\langle e_0, \dots, e_k \rangle)$ . But

$$\begin{aligned} dd\langle e_0, \dots, e_k \rangle &= d\left(\sum_i (-1)^i \langle e_0, \dots, \widehat{e}_i, \dots, e_k \rangle\right) \\ &= \sum_i (-1)^i d\langle e_0, \dots, \widehat{e}_i, \dots, e_k \rangle \\ &= \sum_i (-1)^i \left( \sum_{j < i} (-1)^j \langle e_0, \dots, \widehat{e}_j, \dots, \widehat{e}_i, \dots, e_k \rangle \right. \\ &\quad \left. + \sum_{j > i} (-1)^{j-1} \langle e_0, \dots, \widehat{e}_i, \dots, \widehat{e}_j, \dots, e_k \rangle \right) \\ &= \sum_{0 \leq j < i \leq k} ((-1)^{i+j} + (-1)^{i+j-1}) \langle e_0, \dots, \widehat{e}_j, \dots, \widehat{e}_i, \dots, e_k \rangle = 0. \end{aligned}$$

□

**The algebraic notion of a chain complex.** The following notion is central in what follows.

DEFINITION 1.5. A *chain complex* is a collection of abelian groups indexed by the integers,  $\{C_k\}_{k \in \mathbb{Z}}$ , which is connected by homomorphisms

$$C_\bullet : \dots \rightarrow C_{k+1} \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} C_{k-1} \rightarrow \dots$$

in such a manner that  $d_k d_{k+1} = 0$  for all  $k \in \mathbb{Z}$ .

This definition gives rise to some terminology and notation. Given such a complex  $C_\bullet$ , then an element  $c \in C_k$  is called a  $k$ -cycle of  $C_\bullet$  if  $d_k(c) = 0$  resp. a  $k$ -boundary if  $c \in \text{Im}(d_{k+1})$ . Since  $d_k d_{k+1} = 0$ , a  $k$ -boundary is always a  $k$ -cycle. We write  $Z_k(C_\bullet)$  for  $\text{Ker}(d_k)$  and  $B_k(C_\bullet)$  for  $d_{k+1}(C_{k+1})$  so that  $B_k(C_\bullet) \subset Z_k(C_\bullet)$ . The  $k$ -th homology group of  $C_\bullet$  is then defined by:

$$H_k(C_\bullet) := Z_k(C_\bullet) / B_k(C_\bullet) = \text{Ker}(d_k) / \text{Im}(d_{k+1});$$

elements of which are called *homology classes*. So a  $k$ -cycle  $c \in Z_k(C_\bullet)$  determines a homology class, often briefly referred to as the *class of  $c$* , and denoted  $[c] \in H_k(C_\bullet)$ . Two cycles  $c, c' \in Z_k(C_\bullet)$  have the same class precisely when their difference is a boundary, in which case we also say that  $c$  and  $c'$  are *homologous*.

We say that the complex is *exact at  $C_k$*  if  $H_k(C_\bullet) = 0$ , i.e., if  $d_{k+1}(C_{k+1}) = \text{Ker}(d_k)$ . As this property only depends on the subdiagram  $C_{k+1} \rightarrow C_k \rightarrow C_{k-1}$ , we may freely transfer this notion to the situation where only that subdiagram is

defined: so if  $A \rightarrow B \rightarrow C$  is a diagram of homomorphisms of abelian groups, then we call it exact if the image of  $A \rightarrow B$  equals the kernel of  $B \rightarrow C$ .

We say that the complex  $C_\bullet$  is *exact* if it is exact at every  $C_k$  and we call it *cyclic* if it is exact at every  $C_k$  with  $k \neq 0$ .

**Homology of spaces.** Proposition 1.4 may now be stated as saying that

$$C_\bullet(X) : \cdots \xrightarrow{d} C_{k+1}(X) \xrightarrow{d} C_k(X) \xrightarrow{d} C_{k-1}(X) \xrightarrow{d} \cdots \rightarrow C_0(X) \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

is a chain complex. We call this the *singular chain complex* of  $X$ . We then write  $Z_k(X)$ ,  $B_k(X)$ ,  $H_k(X)$  instead of  $Z_k(C_\bullet(X))$ ,  $B_k(C_\bullet(X))$ ,  $H_k(C_\bullet(X))$  and we refer to their elements as (*singular*)  $k$ -cycles,  $k$ -boundaries,  $k$ -homology classes. The group  $H_k(X)$  will be called the  $k$ -th *singular homology group* of  $X$ . For  $k < 0$  we have  $Z_k(X) = B_k(X) = 0$ , and hence  $H_k(X) = 0$ . For  $k \geq 0$ ,  $Z_k(X)$  and  $B_k(X)$  can be huge (for  $\mathbb{R}$  they are of uncountable rank), but we will see that the homology of reasonable spaces  $X$  is, if not already finitely generated, then at least countably generated. If the homology of  $X$  is finitely generated, then we have for any  $k \in \mathbb{Z}$  defined a number of interest, the rank of  $H_k(X)$  and called the  $k$ -th *Betti number* of  $X$ . We shall see that many spaces can already be distinguished by comparing their Betti numbers.

**Chain maps.** Let  $C'_\bullet$  and  $C_\bullet$  be chain complexes.

**DEFINITION 1.6.** A *chain map* from  $C'_\bullet$  to  $C_\bullet$  is a collection of homomorphisms  $\{\phi_k : C'_k \rightarrow C_k\}_{k \in \mathbb{Z}}$  (often abbreviated as  $\phi_\bullet : C'_\bullet \rightarrow C_\bullet$ ) which commute with the boundary homomorphisms in the sense that for all  $k \in \mathbb{Z}$  we have  $d_k \phi_k = \phi_{k-1} d_k$ .

Notice that in this case,  $\phi_k$  maps  $Z_k(C'_\bullet)$  to  $Z_k(C_\bullet)$  and  $B_k(C'_\bullet)$  to  $B_k(C_\bullet)$ . One of the standard homomorphism theorems then implies that  $\phi_k$  induces a homomorphism  $H_k(C'_\bullet) \rightarrow H_k(C_\bullet)$ . We denote this homomorphism by  $H_k(\phi)$ . The composite of two chain maps  $C''_\bullet \rightarrow C'_\bullet \rightarrow C_\bullet$  is evidently again a chain map (thus, we have a category of chain complexes and chain maps).

According to 1.3, a continuous map  $f : X \rightarrow Y$  defines a chain map  $f_* : C_\bullet(X) \rightarrow C_\bullet(Y)$ . So for every  $k \in \mathbb{Z}$  we have an associated homomorphism  $H_k(X) \rightarrow H_k(Y)$ . We denote this homomorphism  $H_k(f)$ , but if there is little chance of confusion we prefer to write  $f_*$ . (Proposition 1.2 can now be amplified by saying that the formation of the chain complex of a space is a covariant functor from the category of spaces and continuous maps to the category of chain complexes and chain maps.)

In the special case of a chain map that is an inclusion we speak of a (*chain*) *subcomplex*. So this amounts to a collection of subgroups  $\{C'_k \subset C_k\}_{k \in \mathbb{Z}}$  with the property that for all  $k \in \mathbb{Z}$ , we have  $d_k(C'_k) \subset C'_{k-1}$ . Then  $C'_\bullet$  is obviously a chain complex. But we can now also form the *quotient complex*  $\bar{C}_\bullet := C_\bullet / C'_\bullet$ : one of the homomorphism theorems implies that then  $d_k$  induces a homomorphism  $\bar{d}_k : \bar{C}_k \rightarrow \bar{C}_{k-1}$  and from  $d_k d_{k+1} = 0$  it is immediate that  $\bar{d}_k \bar{d}_{k+1} = 0$ .

**Homology of a pair.** This situation occurs for a *space pair*  $(X, A)$ , which we simply define as a space  $X$  together with a subspace  $A$ . (This comprises the case of a single space  $X$ , for that is considered to be space pair by taking for  $A$  the empty

set.) Clearly,  $C_\bullet(A)$  is then a subcomplex of  $C_\bullet(X)$ . We then define the singular chain complex of the pair  $(X, A)$  as the corresponding quotient complex:

$$C_\bullet(X, A) := C_\bullet(X)/C_\bullet(A).$$

The homology of this complex is by definition the *singular homology* of  $(X, A)$ :

$$H_k(X, A) := H_k(C_\bullet(X, A)).$$

Notice that  $C_\bullet(X, \emptyset) = C_\bullet(X)$  and  $H_k(X, \emptyset) = H_k(X)$ .

If  $(X, A)$  and  $(Y, B)$  are space pairs, then by a *map of space pairs*  $f : (X, A) \rightarrow (Y, B)$  we simply mean a (continuous) map  $f : X \rightarrow Y$  with  $f(A) \subset B$ . It follows from the definitions and 1.3 that for such a map  $f_*$  determines a chain map  $C_\bullet(X, A) \rightarrow C_\bullet(Y, B)$  so that for every  $k \in \mathbb{Z}$  we have defined a homomorphism

$$H_k(f) : H_k(X, A) \rightarrow H_k(Y, B).$$

If  $A$  and  $B$  are both empty, then we recover the homomorphism  $H_k(X) \rightarrow H_k(Y)$  found above. We can compose  $f : (X, A) \rightarrow (Y, B)$  with another map  $g : (Y, B) \rightarrow (Z, C)$  of space pairs to form  $gf : (X, A) \rightarrow (Z, C)$  and then find that  $H_k(gf) = H_k(g)H_k(f)$ . We sum up part of this discussion in the guise of a theorem.

**THEOREM 1.7 (0: Functorial character).** *Let  $k \in \mathbb{Z}$ . We have constructed for every space pair  $(X, A)$  an abelian group  $H_k(X, Y)$ , the  $k$ th singular homology group of  $(X, Y)$ , and for every map  $f : (X, A) \rightarrow (Y, B)$  of space pairs a group homomorphism  $H_k(f) : H_k(X, A) \rightarrow H_k(Y, B)$  such that we have a functor from the category of space pairs to the category of abelian groups: the identity map of  $(X, A)$  induces the identity map in  $H_k(X, A)$  and if  $f : (X, A) \rightarrow (Y, B)$  and  $g : (Y, B) \rightarrow (Z, C)$  are maps of space pairs, then  $H_k(gf) = H_k(g)H_k(f)$ .  $\square$*

**PROBLEM 1.** Prove that if  $X$  is arcwise connected, then  $H_0(X) \cong \mathbb{Z}$ .

**PROBLEM 2.** Let  $X_1, \dots, X_n$  be the distinct arc components of the space  $X$ . Prove that  $H_k(X) \cong \bigoplus_{i=1}^n H_k(X_i)$ .

**PROBLEM 3.** Every permutation  $s$  of  $\{0, \dots, k\}$  determines a homeomorphism of  $\Delta^k$  onto itself by  $\langle e_{s(0)}, \dots, e_{s(k)} \rangle$ . Let us regard this homeomorphism as an element of  $C_k(\Delta^k)$ .

(a) Prove that  $\langle e_1, e_0 \rangle + \langle e_0, e_1 \rangle$  is a boundary (so  $k = 1$  here). (Hint: Show first that the constant map  $\Delta^1 \rightarrow \Delta^0$  is a boundary and then consider  $\langle e_0, e_1, e_0 \rangle : \Delta^2 \rightarrow \Delta^1$ .)  
 (b) Prove more generally that  $\langle e_1, e_0, e_2, \dots, e_k \rangle + \langle e_0, e_1, e_2, \dots, e_k \rangle$  is a boundary.  
 (c) Denoting by  $\epsilon(s) \in \{\pm 1\}$  the sign of the permutation  $s$ , show that the difference  $\langle e_{s(0)}, \dots, e_{s(k)} \rangle - \epsilon(s)\langle e_0, \dots, e_k \rangle$  is a boundary.

**PROBLEM 4.** If  $C'_\bullet$  is a subcomplex of a complex  $C_\bullet$ , then show that the inclusion  $d_{k+1}(C_{k+1}) + C'_k \subset d_k^{-1}(C'_{k-1})$  holds. Prove that  $d_k^{-1}(C'_{k-1})/(d_{k+1}(C_{k+1}) + C'_k)$  can be identified with  $H_k(C_\bullet/C'_\bullet)$ .

**PROBLEM 5.** This problem is about the relation between the first homology group and the fundamental group. We call a singular 1-simplex  $\sigma : [e_0, e_1] \rightarrow X$  a *loop* in  $X$  if  $\sigma(e_0) = \sigma(e_1)$ .

(a) Prove that a loop is a 1-cycle.

(b) We call two loops  $\sigma_0, \sigma_1 : [e_0, e_1] \rightarrow X$  (*freely*) *homotopic* if they can be connected by a family of loops  $\{\sigma_s : [e_0, e_1] \rightarrow X\}_{s \in [0,1]}$  which is continuous in the sense that the map  $(s, t) \in [0, 1] \times [0, 1] \mapsto \sigma_s((1-t)e_0 + te_1)$  is so. Prove that being freely homotopic is an equivalence relation.

- (c) Prove that two freely homotopic loops are homologous.  
 (d) Let  $p \in X$ . Prove that the obvious map  $\rho : \pi(X, p) \rightarrow H_1(X)$  is a homomorphism of groups (and hence induces a homomorphism of abelian groups  $\rho_{\text{ab}} : \pi(X, p)_{\text{ab}} \rightarrow H_1(X)$ ).  
 (e) A 1-chain of the form  $\sum_{i \in \mathbb{Z}/r} \sigma_i$  is called a *singular  $r$ -gon* if for all  $i \in \mathbb{Z}/r$ ,  $\sigma_i(e_0) = \sigma_{i-1}(e_1)$ . Prove that such a singular  $r$ -gon is in fact a 1-cycle which is homologous to a loop. (Hint: an  $r$ -gon can be decomposed into 2-simplices.)  
 (f) Prove that  $H_1(X)$  is generated by the classes of loops (hint: use Problem 3).  
 (g) Prove that the homomorphism found in (d) is onto when  $X$  is arcwise connected. (It can be shown that  $\rho_{\text{ab}}$  is always injective. So for arcwise connected  $X$ ,  $\rho_{\text{ab}}$  is always an isomorphism of groups and thus  $H_1(X)$  may then be understood as the abelianized fundamental group.)

**PROBLEM 6.** Let  $f : A \rightarrow B$  be a homomorphism of abelian groups. Prove that  $f$  is injective (resp. surjective) if and only if the sequence  $0 \rightarrow A \xrightarrow{f} B$  is exact at  $A$  (resp.  $A \xrightarrow{f} B \rightarrow 0$  is exact at  $B$ ). (So  $f$  is an isomorphism if and only if the sequence  $0 \rightarrow A \xrightarrow{f} B \rightarrow 0$  is exact.)

## 2. The fundamental properties of singular homology

In this section we derive four important properties of singular homology. We will later find that they in fact characterize this notion and make it possible to compute these groups in practice.

**Homology of a singleton.** This is easily computed.

**THEOREM 2.1 (I: Dimension property).** *The homology of a singleton space  $\{p\}$  is given by  $H_0(\{p\}) = \mathbb{Z}$  and  $H_k(\{p\}) = 0$  for  $k \neq 0$ .*

**PROOF.** For every  $k \geq 0$  there is precisely one singular  $k$ -simplex  $\sigma_k : \Delta^k \rightarrow \{p\}$ . Now  $d\sigma_k$  has  $k + 1$  terms  $\pm\sigma_{k-1}$  (with alternating signs), at least when  $k > 0$ . We find that

$$d\sigma_k = \begin{cases} \sigma_{k-1} & \text{when } k > 0 \text{ is even;} \\ 0 & \text{otherwise,} \end{cases}$$

from which the theorem easily follows.  $\square$

Let  $C'_\bullet$  be a subcomplex of a chain complex  $C_\bullet$ . We define for every  $k \in \mathbb{Z}$  a homomorphism, commonly called the *connecting homomorphism*,

$$\partial_k : H_k(C_\bullet/C'_\bullet) \rightarrow H_{k-1}(C'_\bullet)$$

as follows. According to Problem 4 we can identify  $H_k(C_\bullet/C'_\bullet)$  with the quotient  $d^{-1}C'_{k-1}/(d(C_{k+1}) + C'_k)$ . Thus every  $w \in H_k(C_\bullet/C'_\bullet)$  can be represented by a  $c \in C_k$  with  $d(c) \in C'_{k-1}$ . Since  $dd(c) = 0$ , it follows that  $d(c) \in Z_{k-1}(C'_\bullet)$  (but since  $c$  need not lie in  $C'_k$ , we cannot say that  $d(c) \in B_{k-1}(C'_\bullet)$ ). Our  $\partial_k(w)$  will be the class of  $d(c)$  in  $H_{k-1}(C'_\bullet)$ . This is indeed well-defined, because in case  $c \in d(C_{k+1}) + C'_k$ , then  $d(c) \in dd(C_{k+1}) + d(C'_k) = d(C'_k) = B_{k-1}(C'_\bullet)$  and hence its class in  $H_{k-1}(C'_\bullet)$  is zero.

**An exact sequence.** We begin the algebraic version.

**THEOREM 2.2** (Long exact homology sequence). *Let us denote the inclusion  $C'_\bullet \subset C_\bullet$  by  $\alpha$  and the reduction  $C_\bullet \rightarrow C_\bullet/C'_\bullet$  by  $\beta$ . Then the sequence*

$$\dots \xrightarrow{\partial_{k+1}} H_k(C'_\bullet) \xrightarrow{H_k(\alpha)} H_k(C_\bullet) \xrightarrow{H_k(\beta)} H_k(C_\bullet/C'_\bullet) \xrightarrow{\partial_k} H_{k-1}(C'_\bullet) \xrightarrow{H_{k-1}(\alpha)} \dots$$

is exact: the kernel of any arrow equals the image of the arrow that precedes it. Moreover the homomorphism  $\partial_k$  is ‘natural’ in the following sense: if  $D_\bullet$  is a second complex with subcomplex  $D'_\bullet$  and  $\phi : C_\bullet \rightarrow D_\bullet$  is a chain map with  $\phi(C'_k) \subset D'_k$  for all  $k \in \mathbb{Z}$  (so that  $\phi$  restricts to a chain map  $\phi' : C'_\bullet \rightarrow D'_\bullet$ ), then the diagram

$$\begin{array}{ccc} H_k(C_\bullet/C'_\bullet) & \xrightarrow{\partial_k} & H_{k-1}(C'_\bullet) \\ H_k(\phi) \downarrow & & \downarrow H_k(\phi') \\ H_k(D_\bullet/D'_\bullet) & \xrightarrow{\partial_k} & H_{k-1}(D'_\bullet) \end{array}$$

is commutative.

**PROOF.** Clearly,  $\beta_k \alpha_k = 0$ , and therefore  $H_k(\beta)H_k(\alpha) = 0$ . Similar arguments show that  $\partial_k H_k(\beta) = 0$  and  $H_{k-1}(\alpha)\partial_k = 0$ . In other words, the long sequence is like a complex: the kernel of each of its maps contains the image of its predecessor. We next show the opposite inclusions.

Any  $u \in \text{Ker}(H_k(\alpha))$  can be represented by a  $c' \in C'_k$  which can be written  $c' = d(c)$  for some  $c \in C_{k+1}$ . So then is  $c \in d^{-1}(C'_k)$  and hence the image of  $c$  in  $C_\bullet/C'_\bullet$  is a  $k$ -cycle. It follows from the definition of  $\partial_{k+1}$  that the class  $w \in H_{k+1}(C_\bullet/C'_\bullet)$  of this  $k$ -cycle has the property that  $\partial_{k+1}(w) = u$ .

Any  $v \in \text{Ker}(H_k(\beta))$  can be represented by a  $c \in \text{Ker } d_k$  which can be written  $c = d(\hat{c}) + c'$  for some  $\hat{c} \in C_{k+1}$  and  $c' \in C'_k$ . Then  $v$  is also represented by  $c' = c - d(\hat{c}) \in C'_k$ . Since  $d(c') = d(c) = 0$ ,  $c'$  defines a homology class  $u \in H_k(C'_\bullet)$ . Clearly,  $H_k(\alpha)(u) = v$ .

Any  $w \in \text{Ker}(\partial_k)$  can be represented by a  $c \in C_k$  with the property that  $d(c) = d(c')$  for some  $c' \in C'_k$ . Then  $w$  is also represented by  $c - c'$ . Since  $d(c - c') = 0$ ,  $c - c'$  defines a homology class  $v \in H_k(C_\bullet)$ . Clearly,  $H(\beta)(v) = w$ .

The naturality of  $\partial_k$  follows easily from its definition.  $\square$

A *space triple* consists of a space  $X$ , a subspace  $Y$  of  $X$  and subspace  $Z$  of  $Y$  (and is then written  $(X, Y, Z)$ ). Such a triple gives rise to two maps (actually inclusions) of space pairs

$$i : (Y, Z) \rightarrow (X, Z), \quad j : (X, Z) \rightarrow (X, Y).$$

These induce chain maps. Notice that  $C_k(Y, Z) = C_k(Y)/C_k(Z)$  can be regarded as a subcomplex of  $C_k(X)/C_k(Z) = C_k(X, Z)$  and that the quotient complex is  $C_k(X)/C_k(Y) = C_k(X, Y)$ . So from Theorem 2.2 we conclude:

**THEOREM 2.3 (II: Long exact sequence for a triple).** *The sequence*

$$\dots \xrightarrow{\partial_{k+1}} H_k(Y, Z) \xrightarrow{i_k} H_k(X, Z) \xrightarrow{j_k} H_k(X, Y) \xrightarrow{\partial_k} H_{k-1}(Y, Z) \xrightarrow{i_{k-1}} H_{k-1}(X, Z) \xrightarrow{j_{k-1}} \dots,$$

is exact. Moreover, the homomorphism  $\partial_k$  is ‘natural’ in the following sense: if  $f : (X, Y, Z) \rightarrow (X', Y', Z')$  is a map of topological triples (so  $X \xrightarrow{f} Y$  is continuous, sends



$Y$  to  $Y'$  and  $Z$  to  $Z'$ ), then the diagram

$$\begin{array}{ccc} H_k(X, Y) & \xrightarrow{\partial_k} & H_{k-1}(Y, Z) \\ \downarrow & & \downarrow \\ H_k(X', Y') & \xrightarrow{\partial_k} & H_{k-1}(Y', Z') \end{array}$$

is commutative (the vertical maps are induced by restrictions of  $f$ ).

For  $Z = \emptyset$  we get the exact sequence of a space pair:

$$\cdots \xrightarrow{\partial_{k+1}} H_k(Y) \xrightarrow{i_k} H_k(X) \xrightarrow{j_k} H_k(X, Y) \xrightarrow{\partial_k} H_{k-1}(Y) \xrightarrow{i_{k-1}} H_{k-1}(X) \xrightarrow{j_{k-1}} \cdots$$

**Homotopy.** The third property of singular homology worth isolating is:

**THEOREM 2.4 (III: Homotopy invariance).** For a space pair  $(X, A)$  the injections (of pairs)  $i^0, i^1 : (X, A) \rightarrow (X \times [0, 1], A \times [0, 1])$ ,  $i^\alpha(x) = (x, \alpha)$ , yield the same map on homology.

Before we begin the proof proper, let us have a look at the ‘prism’  $\Delta^k \times [0, 1]$  (viewing it as a subspace of  $\mathbb{R}^{k+1} \times \mathbb{R} = \mathbb{R}^{k+2}$ ). If we write  $e_i^t$  for  $(e_i, t)$ , then we observe that this prism can be decomposed in the  $(k+1)$ -simplices

$$\{[e_0^0, \dots, e_j^0, e_j^1, \dots, e_k^1]\}_{j=0}^k$$

(check this for  $k=1$  and  $k=2$ ). This might suggest to consider on  $\Delta^k \times [0, 1]$  the  $(k+1)$ -chain

$$P^{k+1} := \sum_{j=0}^k (-1)^j \langle e_0^0, \dots, e_j^0, e_j^1, \dots, e_k^1 \rangle.$$

(the signs have been chosen in such a manner that no ‘interior’  $k$ -simplex appears in the support of  $dP^{k+1}$ ). For any singular  $k$ -simplex  $\sigma : \Delta^k \rightarrow X$  we now define a  $(k+1)$ -chain  $P(\sigma)$  on  $X \times [0, 1]$  by

$$P(\sigma) = (\sigma \times \mathbf{1}_{[0,1]})_* P^{k+1},$$

where  $\sigma \times \mathbf{1}_{[0,1]} : (p, t) \mapsto (\sigma(p), t)$ . We extend  $P$  additively to a homomorphism

$$P : C_k(X) \rightarrow C_{k+1}(X \times [0, 1]), \quad \sum_i c_i \sigma_i \mapsto \sum_i c_i P(\sigma_i) \quad (k = 0, 1, 2, \dots).$$

For  $k < 0$  we let  $P$  be the zero map.

**LEMMA 2.5.** For every map  $f : X \rightarrow Y$  of spaces and  $k \in \mathbb{Z}$  we have

$$Pf_k = (f \times \mathbf{1}_{[0,1]})_{k+1} P.$$

**PROOF.** For a singular  $k$ -simplex  $\sigma : \Delta^k \rightarrow X$ , we compute

$$\begin{aligned} Pf_k(\sigma) &= P(f\sigma) = (f\sigma \times \mathbf{1}_{[0,1]})_{k+1}(P^{k+1}) \\ &= (f \times \mathbf{1}_{[0,1]})_{k+1}(\sigma \times \mathbf{1}_{[0,1]})_{k+1}(P^{k+1}) \\ &= (f \times \mathbf{1}_{[0,1]})_{k+1} P(\sigma). \end{aligned}$$

(We used that  $f\sigma \times \mathbf{1}_{[0,1]} = (f \times \mathbf{1}_{[0,1]})(\sigma \times \mathbf{1}_{[0,1]})$ .) □

If  $(X, A)$  is a space pair, then  $P : C_k(X) \rightarrow C_{k+1}(X \times [0, 1])$  maps the subgroup  $C_k(A)$  to  $C_{k+1}(A \times [0, 1])$  and so  $P$  determines a homomorphism

$$C_k(X, A) \rightarrow C_{k+1}(X \times [0, 1], A \times [0, 1]).$$

We continue to denote this by  $P$ .

LEMMA 2.6.  $dP + Pd = i_*^1 - i_*^0$ .

PROOF. It suffices to check this for every singular  $k$ -simplex  $\sigma : \Delta^k \rightarrow X$ . Because of 1.3 and 2.5 we have

$$(2) \quad (dP + Pd)(\sigma) = (dP + Pd)\sigma_* \langle e_0, \dots, e_k \rangle = (\sigma \times \mathbf{1}_{[0,1]})_* (dP + Pd) \langle e_0, \dots, e_k \rangle.$$

Furthermore

$$(3) \quad i_*^\alpha(\sigma) = (\sigma \times \mathbf{1}_{[0,1]})_* \langle e_0^\alpha, \dots, e_k^\alpha \rangle.$$

We next expand:

$$\begin{aligned} dP \langle e_0, \dots, e_k \rangle &= \sum_j (-1)^j d \langle e_0^0, \dots, e_j^0, e_j^1, \dots, e_k^1 \rangle \\ &= \sum_{i \leq j} (-1)^{i+j} \langle e_0^0, \dots, \widehat{e}_i^0, \dots, e_j^0, e_j^1, \dots, e_k^1 \rangle \\ &\quad + \sum_{i \geq j} (-1)^{i+j+1} \langle e_0^0, \dots, e_j^0, e_j^1, \dots, \widehat{e}_i^1, \dots, e_k^1 \rangle. \\ Pd \langle e_0, \dots, e_k \rangle &= \sum_i (-1)^i P \langle e_0, \dots, \widehat{e}_i \dots e_k \rangle \\ &= \sum_{j < i} (-1)^{i+j} \langle e_0^0, \dots, e_j^0, e_j^1, \dots, \widehat{e}_i^1, \dots, e_k^1 \rangle \\ &\quad + \sum_{j > i} (-1)^{i+j-1} \langle e_0^0, \dots, \widehat{e}_i^0, \dots, e_j^0, e_j^1, \dots, e_k^1 \rangle. \end{aligned}$$

Addition of both expressions makes all terms on the right with  $i \neq j$  cancel each other:

$$\begin{aligned} (dP + Pd) \langle e_0, \dots, e_k \rangle &= \sum_i (\langle e_0^0, \dots, e_{i-1}^0, e_i^1, \dots, e_k^1 \rangle - \langle e_0^0, \dots, e_i^0, e_{i+1}^1, \dots, e_k^1 \rangle) \\ &= \langle e_0^1, \dots, e_k^1 \rangle - \langle e_0^0, \dots, e_k^0 \rangle \end{aligned}$$

The lemma then follows from the last identity combined with the formulae (2) and (3).  $\square$

PROOF OF 2.4. If  $c \in Z_k(X, A)$ , then it follows from the last lemma that

$$(4) \quad i_*^1(c) - i_*^0(c) = dP(c) + Pd(c) = dP(c)$$

So  $i_*^1(c)$  and  $i_*^0(c)$  have the same class in  $H_k(X \times [0, 1], A \times [0, 1])$ .  $\square$

This theorem spawns some discussion. We call two maps between space pairs  $f, f' : (X, A) \rightarrow (Y, B)$  *homotopic* if there is a map  $F : (X \times [0, 1], A \times [0, 1]) \rightarrow (Y, B)$  with  $F(x, 0) = f(x)$  and  $F(x, 1) = f'(x)$ ; the map  $F$  is then called a *homotopy* from  $f$  to  $f'$ . We usually regard this as a (continuous) path of maps  $F_t : (X, A) \rightarrow (Y, B)$ ,  $F_t(x) = F(x, t)$  which connects  $f = F_0$  with  $f' = F_1$ .

COROLLARY 2.7. *Homotopic maps of space pairs  $f, f' : (X, A) \rightarrow (Y, B)$  induce the same maps on homology.*

PROOF. If  $F$  is a homotopy from  $f$  to  $f'$ , then  $f = F_i^0$  and  $f' = F_i^1$ . Now apply Theorems 1.7 and 2.4.  $\square$

'Being homotopic' is an equivalence relation: reflexivity is obvious (take  $F(x, t) = f(x)$ ) and so is the symmetry (pass from  $F(x, t)$  to  $F(x, 1 - t)$ ). As to transitivity: if  $F$  resp.  $F'$  is a homotopy from  $f$  to  $f'$  resp. from  $f'$  to  $f''$ , then

$$F''(x, t) := \begin{cases} F(x, 2t) & \text{als } 0 \leq t \leq \frac{1}{2}; \\ F'(x, 2t - 1) & \text{als } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a homotopy from  $f$  to  $f''$ . The equivalence class of  $f$  is called its *homotopy class* and we denote that class by  $[f]$  (it is essentially the arc component of  $f$  in a space of maps). Homotopy classes can be composed: if  $f, f' : (X, A) \rightarrow (Y, B)$  are connected by a homotopy  $F$ , and  $g, g' : (Y, B) \rightarrow (Z, C)$  by a homotopy  $G$ , then  $H(x, t) := G(F(x, t), t)$  is a homotopy from  $gf$  to  $g'f'$ . So  $[gf]$  only depends on  $[f]$  and  $[g]$ , so that we may also write this as  $[g][f]$ . (Thus is defined a *category* with objects space pairs and morphisms homotopy classes. Corollary 2.7 tells us that 'taking the  $k$ -th homology group' is a functor from this category to the category of abelian groups.) In this setting the notion of isomorphism is that of a *homotopy equivalence of pairs*: a map of space pairs  $f : (X, A) \rightarrow (Y, B)$  is thus called if it has a *homotopy inverse*  $g : (Y, B) \rightarrow (X, A)$ . This means that  $gf$  resp.  $fg$  is homotopic to the de identity of  $(X, A)$  resp.  $(Y, B)$ . We then say that the space pairs  $(X, A)$  and  $(Y, B)$  are *homotopy equivalent*. Since we regard a single space  $X$  as the space pair  $(X, \emptyset)$ , this entails corresponding notions for spaces rather than space pairs.

A space is called *contractible* if it is homotopy equivalent to a singleton, more precisely, if it satisfies one of the equivalent conditions in Lemma 2.8 below (the proof of which is left as an exercise).

LEMMA 2.8. *For a space  $X$  the following are equivalent:*

- (i)  $X$  is nonempty and if  $\{o\}$  is a singleton, then the unique map  $f : X \rightarrow \{o\}$  is a homotopy equivalence,
- (ii)  $X$  is nonempty and if  $p \in X$ , then the inclusion  $\{p\} \subset X$  is a homotopy equivalence.

For instance,  $\mathbb{R}^n$  is contractible. On the other hand, the empty set is not contractible (why?). We shall see that no sphere is contractible. The next corollary shows that the homology groups are homotopy invariants.

COROLLARY 2.9. *A homotopy equivalence  $f : (X, A) \rightarrow (Y, B)$  induces an isomorphism on homology. In particular, if  $X$  is contractible and  $p \in X$ , then  $H_k(X, \{p\}) = 0$  and  $H_k(X) = H_k(\{p\})$  for all  $k \in \mathbb{Z}$ .*

PROOF. If  $g : (Y, B) \rightarrow (X, A)$  is a homotopy inverse, then  $g_k f_k$  resp.  $f_k g_k$  give the identity in  $H_k(X, A)$  resp.  $H_k(Y, B)$ . Hence  $g_k$  is a two sided inverse of  $f_k$ .  $\square$

The very definition of the homology of a space pair  $(X, A)$  seems to suggest that the internal structure of  $A$  hardly matters. To some extent, that is the case, witness the following *excision theorem*.

THEOREM 2.10 (IV: Excision). *If  $(X, A)$  is a space pair and  $B \subset A$  is a subspace  $\overline{B} \subset A^\circ$ , then the homomorphisms  $H_k(X - B, A - B) \rightarrow H_k(X, A)$  (induced by the inclusion  $(X - B, A - B) \subset (X, A)$ ) are in fact isomorphisms.*

This is intimately related to:

**THEOREM 2.11 (Mayer-Vietoris).** *Let a space  $X$  be the union of two open subsets  $U_0$  and  $U_1$  and denote by  $i_\alpha : U_0 \cap U_1 \subset U_\alpha$  and  $j_\alpha : U_\alpha \subset X$  the inclusions. Then we have a ‘natural’ exact sequence*

$$\cdots \rightarrow H_k(U_0 \cap U_1) \xrightarrow{(-i_{0*}, i_{1*})} H_k(U_0) \oplus H_k(U_1) \xrightarrow{j_{0*} + j_{1*}} H_k(X) \xrightarrow{\partial_k} H_{k-1}(U_0 \cap U_1) \rightarrow \cdots$$

Both theorems will be derived from a single assertion. Let  $X$  be a topological space and  $\mathcal{U}$  an open covering of  $X$ . Denote by  $C_k(X; \mathcal{U})$  the set of  $k$ -chains of  $X$  whose support is *subordinate to  $\mathcal{U}$*  in the sense that it consists of singular simplices whose image lies in a member of  $\mathcal{U}$ . This gives a subcomplex  $C_\bullet(X; \mathcal{U}) \subset C_\bullet(X)$ ; in fact,

$$C_\bullet(X; \mathcal{U}) = \sum_{U \in \mathcal{U}} C_\bullet(U).$$

(The right hand side stands for finite sums of which each term lies in some  $C_\bullet(U)$ ,  $U \in \mathcal{U}$ .) For a subspace  $A \subset X$ , it is clear that  $C_\bullet(X; \mathcal{U}) \cap C_\bullet(A) = C_\bullet(A; \mathcal{U})$ . We thus also obtain a subcomplex

$$C_\bullet(X, A; \mathcal{U}) := C_\bullet(X; \mathcal{U}) / C_\bullet(A; \mathcal{U}) \subset C_\bullet(X) / C_\bullet(A) = C_\bullet(X, A).$$

Notice that  $C_\bullet(X, A; \mathcal{U}) = \sum_{U \in \mathcal{U}} C_\bullet(U, U \cap A)$ .

**PROPOSITION 2.12.** *The inclusion  $C_\bullet(X, A; \mathcal{U}) \subset C_\bullet(X, A)$  yields an isomorphism on all homology groups.*

**PROOF THAT PROPOSITION 2.12 IMPLIES THEOREM 2.10.** Consider the open covering  $\mathcal{U}$  of  $X$  consisting of the two open parts  $A^\circ$  and  $X - \overline{B}$  and the commutative square

$$\begin{array}{ccc} C_\bullet(X - B, A - B; \mathcal{U}) & \subset & C_\bullet(X - B, A - B) \\ \downarrow & & \downarrow \\ C_\bullet(X, A; \mathcal{U}) & \subset & C_\bullet(X, A). \end{array}$$

According to 2.12 the horizontal inclusions yield isomorphisms on homology. The first vertical map is in fact an isomorphism, for the obvious map

$$C_k(X - \overline{B}) / C_k(A - \overline{B}) \rightarrow (C_k(X - \overline{B}) + C_k(A^\circ)) / (C_k(A - \overline{B}) + C_k(A^\circ))$$

is one (we used that if  $G$  and  $K$  are subgroups of some unnamed abelian group and  $H$  is a subgroup  $G$  which contains  $G \cap K$ , then the obvious map  $G/H \rightarrow (G + K)/(H + K)$  is an isomorphism) and we recognize the right hand side as  $C_k(X, A; \mathcal{U})$  and note that the left hand side does not change if we substitute  $(X - B, A - B)$  for  $(X, A)$ , so that is also maps isomorphically to  $C_k(X - B, A - B; \mathcal{U})$ . As now three of the four sides of our commutative square yield isomorphisms on homology, so will the remaining side.  $\square$

**PROOF THAT PROPOSITION 2.12 IMPLIES THEOREM 2.11.** Consider the open covering  $\mathcal{U}$  of  $X$  consisting of the two open subsets  $U_0$  and  $U_1$ . Then  $C_\bullet(X; \mathcal{U}) = C_\bullet(U_0) + C_\bullet(U_1)$  (where both members are regarded as subcomplexes of  $C_\bullet(X)$ ). A (rather trivial) result from algebra says that for two subgroups  $G_0, G_1$  of an abelian group  $G$ ,  $G_0 + G_1$  can be regarded as a quotient of  $G_0 \oplus G_1$  by the (antidiagonally embedded) subgroup  $G_0 \cap G_1$  via  $g \in G_0 \cap G_1 \mapsto (-g, g) \in G_0 \oplus G_1$ . So  $C_\bullet(X; \mathcal{U})$  can be regarded as the quotient complex of  $C_\bullet(U_0) \oplus C_\bullet(U_1)$  by the subcomplex  $C_\bullet(U_0) \cap C_\bullet(U_1) = C_\bullet(U_0 \cap U_1)$ . The theorem now follows from the long exact homology sequence and the fact that the homology of  $C_\bullet(X; \mathcal{U})$  equals the homology of  $X$ .  $\square$

The idea behind the proof of 2.12 is to replace a one term chain  $\sigma$  by a chain  $\sigma_*(c)$ , whose support is subordinate to  $\sigma^{-1}\mathcal{U}$ . We do this by means of a repeated barycentric subdivision. For a 1-simplex this is simply the partition into two equal parts, for a 2-simplex it is the division into triangles by means of medians. Notice that the vertices of such a triangle consist of a vertex of the original simplex, the midpoint of a face and the barycenter of the original 2-simplex. If we subsequently go up to dimension three, then it becomes plausible that the refinement can be carried out inductively as follows: given a  $k$ -simplex  $s$ , then the vertices of a  $k$ -simplex of its *barycentric subdivision* consist of the barycenter  $z(s)$  of  $s$  and the  $k$  vertices of a  $(k-1)$ -simplex of the barycentric subdivision of a  $(k-1)$ -dimensional facet of  $s$ . (We recall that the *barycenter*  $z(s)$  of a  $k$ -simplex  $s = [p_0, \dots, p_k]$  is the unique point all of whose barycentric coordinates are equal:  $z(s) = \sum_{i=0}^k \frac{1}{k+1} p_i$ .) We shall need the following elementary result.

LEMMA 2.13. *The diameter of a simplex (=length of its longest edge) of the barycentric subdivision of a  $k$ -simplex  $s \subset \mathbb{R}^n$  is  $\frac{k}{k+1}$  times the diameter of  $s$  at most.*

PROOF. With induction on  $k$ . If  $s = [p_0, \dots, p_k]$ , then

$$z(s) - p_j = \sum_i \frac{p_i - p_j}{k+1},$$

and hence has length  $\leq \sum_i \frac{1}{k+1} \text{dist}(p_i, p_j) \leq \frac{k}{k+1} \text{diam}(s)$ . The remaining lengths to be estimated are by induction hypothesis  $\leq \frac{k-1}{k} \text{diam}(s')$ , with  $s'$  a face of  $s$ , and hence certainly of diameter  $\leq \frac{k}{k+1} \text{diam}(s)$ .  $\square$

COROLLARY 2.14. *Let  $X$  be a space and  $\mathcal{U}$  an open covering of  $X$ . Then for every singular simplex  $\sigma : \Delta^k \rightarrow X$  there exist an  $n \in \mathbb{N}$ , such that  $\sigma$  maps every simplex of the  $n$ -th barycentric subdivision in a member of  $\mathcal{U}$ .*

PROOF. Observe that  $\sigma^{-1}\mathcal{U}$  is an open covering of the compact metric space  $\Delta^k$ . As is well-known, there exists then an  $\epsilon > 0$  such that every  $\epsilon$ -ball in  $\Delta^k$  is contained in a member of  $\sigma^{-1}\mathcal{U}$ . Choose  $n$  so large that  $(\frac{k}{k+1})^n < \epsilon$ .  $\square$

We do an analogous construction for chains. First we stipulate that  $B^0 := \langle e_0 \rangle$ . Suppose now that we have defined with induction on  $k > 0$  a  $(k-1)$ -chain  $B^{k-1}$  on  $\Delta^{k-1}$  whose support consists of members of the barycentric subdivision of  $\Delta^{k-1}$ . We turn  $B^{k-1}$  into an operator  $\beta_{k-1}$  on chain complexes by letting it assign to every singular  $(k-1)$ -simplex  $\sigma : \Delta^{k-1} \rightarrow X$  the chain  $\beta_{k-1}(\sigma) := \sigma_*(B^{k-1})$  and extend this linearly to a homomorphism:

$$\begin{aligned} \beta_{k-1} : C_{k-1}(X) &\rightarrow C_{k-1}(X), \\ \sum c_\sigma \sigma &\mapsto \sum c_\sigma \sigma_*(B^{k-1}). \end{aligned}$$

The induction step is set by

$$(5) \quad B^k := \langle z(\Delta^k), \beta_{k-1} d \langle e_0, \dots, e_k \rangle \rangle,$$

where the notation should be self-evident: an expression  $\langle z, \langle p_0, \dots, p_l \rangle \rangle$  stands for  $\langle z, p_0, \dots, p_l \rangle$ .

The inductive definition for  $B^k$  entails an inductive definition for the operator  $\beta_k$  on  $C_k(X)$  for every space  $X$  and every  $k$ . Clearly:

LEMMA 2.15. *For every map between spaces  $f : X \rightarrow Y$ , we have that  $\beta_k f_* = f_* \beta_k$  for all  $k$ .*  $\square$

LEMMA 2.16. *The maps  $\beta_k$  make up a chain map: we have  $d\beta_k = \beta_{k-1}d$ .*

PROOF. This is trivial for 0-chains. We proceed with induction: we assume the identity proved for  $(k-1)$ -chains and prove it then for  $k$ -chains. By a now familiar argument we only need to do this for  $\langle e_0, \dots, e_k \rangle$ . We have

$$\begin{aligned} d\beta_k \langle e_0, \dots, e_k \rangle &= d(B^k) = d\langle z(\Delta^k), \beta_{k-1}d \langle e_0, \dots, e_k \rangle \rangle \\ &\quad \text{(by the inductive definition)} \\ &= \beta_{k-1}d \langle e_0, \dots, e_k \rangle - \langle z(\Delta^k), d\beta_{k-1}d \langle e_0, \dots, e_k \rangle \rangle \\ &= \beta_{k-1}d \langle e_0, \dots, e_k \rangle - \langle z(\Delta^k), \beta_{k-2}dd \langle e_0, \dots, e_k \rangle \rangle \\ &\quad \text{(by the induction hypothesis)} \\ &= \beta_{k-1}d \langle e_0, \dots, e_k \rangle. \end{aligned}$$

$\square$

The preceding lemma implies that  $\beta$  sends cycles to cycles, and boundaries to boundaries. Thus  $\beta$  determines an endomorphism of the homology groups. We shall see that this endomorphism is the identity; in fact we shall prove a stronger property.

LEMMA 2.17. *For every space  $X$  and integer  $k$  there exist a homomorphism*

$$I_k : C_k(X) \rightarrow C_{k+1}(X)$$

*with the property that  $I_{k-1}d_k + d_{k+1}I_k = \beta_k - \mathbf{1} : C_k(X) \rightarrow C_k(X)$  and such that for every map of spaces,  $f : X \rightarrow Y$ ,  $f_*I_k = I_k f_*$ .*

PROOF. For  $k \leq 0$  we take  $I_k = 0$ . Since  $\beta_0$  is the identity for all  $X$ , it follows that both properties are satisfied. We proceed with induction and assume that  $I_l$  has been constructed for  $l < k$  ( $k > 0$ ). In order to define  $I_k$ , we first notice that the image of  $\beta_k - \mathbf{1} - I_{k-1}d$  consists of cycles:

$$\begin{aligned} d(\beta_k - \mathbf{1} - I_{k-1}d) &= d\beta_k - d - (dI_{k-1})d \\ &= d\beta_k - d - (-I_{k-2}d + \beta_{k-1} - \mathbf{1})d \\ &= (d\beta_k - \beta_{k-1}d) + I_{k-2}dd = 0. \end{aligned}$$

In particular,  $a := (\beta_k - \mathbf{1} - I_{k-1}d) \langle e_0, \dots, e_k \rangle$  is a  $k$ -cycle on  $\Delta^k$ . Since  $\Delta^k$  is star shaped,  $\Delta^k$  has no homology in dimension  $k$ , and so  $a$  is even a boundary:  $a = d\hat{a}$  for a chain  $\hat{a}$  on  $\Delta^k$ . We define  $I_k(X)$  by means of the familiar procedure: for a singular  $k$ -simplex  $\sigma : \Delta^k \rightarrow X$  we put  $I_k(\sigma) := \sigma_*(\hat{a})$ , after which we extend  $I_k$  in the usual manner to a homomorphism  $C_k(X) \rightarrow C_{k+1}(X)$ . The first property now holds by construction for  $\langle e_0, \dots, e_k \rangle$ :

$$\begin{aligned} (I_{k-1}d + dI_k) \langle e_0, \dots, e_k \rangle &= (I_{k-1}d \langle e_0, \dots, e_k \rangle + d\hat{a}) \\ &= (I_{k-1}d \langle e_0, \dots, e_k \rangle + a) = (\beta_k - \mathbf{1}) \langle e_0, \dots, e_k \rangle \end{aligned}$$

and then follows in general by the familiar argument. The second property follows easily from the definition.  $\square$

**PROOF OF PROPOSITION 2.12.** *Surjectivity:* Let  $u \in H_k(X, A)$  be arbitrary. Choose a representative  $c \in C_k(X)$  (so with  $d(c) \in C_{k-1}(A)$ ), and apply Corollary 2.14 to the open covering  $\mathcal{U}$  and the (finitely many) singular simplices in the support of  $c$ . We then find that there exists a  $n \in \mathbb{N}$  such that the support of  $\beta^n(c)$  is subordinate to  $\mathcal{U}$ . According to Lemma 2.17 we have  $c - \beta(c) = dI(c) + Id(c)$ . Since  $d(c) \in C_{k-1}(A)$ , the right hand side is contained in  $B_k(X) + C_k(A)$ , and hence  $u$  is also represented by  $\beta(c)$ . With induction we find that  $u$  also represented by  $\beta^n(c)$ . The latter projects to a  $k$ -cycle of  $C_\bullet(X, A; \mathcal{U})$  and so it follows that  $C_\bullet(X, A; \mathcal{U}) \subset C_\bullet(X, A)$  induces a surjection on  $H_k$ .

*Injectivity:* Let  $v \in H_k(C_\bullet(X, A; \mathcal{U}))$  and suppose that its image in  $H_k(X, A)$  is zero. So if we represent  $v$  by  $b \in C_k(X; \mathcal{U})$ , then we must be able to write  $b$  as  $b = d\hat{c} + a$  with  $\hat{c} \in C_{k+1}(X)$  and  $a \in C_k(A)$ . Choose  $n \in \mathbb{N}$  large enough as to make the support of  $\beta^n(\hat{c})$  subordinate to  $\mathcal{U}$ . We have

$$\beta^n(b) = \beta^n d(\hat{c}) + \beta^n(a) = d\beta^n(\hat{c}) + \beta^n(a).$$

from which we read off among other things that the support of  $\beta^n(a) = \beta^n(b) - d\beta^n(\hat{c})$  is subordinate to  $\mathcal{U}$ . But then this identity implies that the image of  $\beta^n(b)$  in  $C_k(X, A; \mathcal{U})$  is contained in  $dC_{k-1}(X, A; \mathcal{U})$ . On the other hand,  $\beta^n(b)$  is also a representative of  $v$  (see the surjectivity proof). Hence  $v = 0$ .  $\square$

**PROBLEM 7.** A subset  $X \subset \mathbb{R}^n$  is called *star shaped* if  $x \in X$  implies that  $tx \in X$  for  $0 \leq t \leq 1$ . Prove that such a subset is contractible.

**PROBLEM 8.** Let  $s$  be a  $k$ -simplex.

- Prove that every vertex of the barycentric subdivision of  $s$  is the barycenter of a facet of  $s$ .
- Prove that for every (strictly) ascending sequence of facets  $s_0 \subset s_1 \subset \cdots \subset s_l$ ,  $l \leq k$ , of  $s$  the barycenters  $z(s_0), \dots, z(s_l)$  are the vertices of a simplex of the barycentric subdivision of  $s$ .
- Prove that conversely, every simplex of the barycentric subdivision of  $s$  is obtained in this manner.

**PROBLEM 9.** Let  $(X, A)$  be a space pair. Prove that  $H_k(X, A) = 0$  for all  $k \leq n$  if and only if the homomorphism  $H_k(A) \rightarrow H_k(X)$  is an isomorphism for all  $k < n$  and surjective for  $k = n$ .

**PROBLEM 10.** Let  $(X, A)$  be a space pair.

- Let  $r : X \rightarrow A$  be a *retraction*, by which we mean a map whose restriction to  $A$  is the identity map. Prove that  $H_k(X) \cong H_k(A) \oplus \text{Ker}(r_k)$ .
- Suppose that there exists a continuous map  $R : X \times [0, 1] \rightarrow X$  such that  $R(a, t) = a$  for all  $a \in A$  and for which  $R(x, 0) = x$  and  $R(x, 1) \in A$  for all  $x \in X$ . (We then call  $A$  a *deformation retract* of  $X$  and  $R$  is called a *deformation retraction*.) Prove that for every subspace  $B \subset A$  the inclusion  $(A, B) \subset (X, B)$  induces isomorphisms on homology.

**PROBLEM 11.** Prove that any map from a space  $X$  to the  $n$ -sphere which is *not* surjective is homotopic to a constant map. (Hint: Use that  $S^n$  minus a point is homeomorphic to  $\mathbb{R}^n$ .)

**PROBLEM 12.** Let  $f : X \rightarrow Y$  be map between spaces. The *mapping cylinder* of  $f$  is the quotient space of the disjoint union of  $X \times [0, 1]$  and  $Y$  obtained by identifying  $(x, 0) \in X \times [0, 1]$  with  $f(x) \in Y$ . We denote it by  $Z_f$ .

(a) Prove that  $X \rightarrow Z_f, x \mapsto (x, 1)$  and the obvious map  $Y \rightarrow Z_f$  map these spaces homeomorphically onto closed subsets of  $Z_f$ . (For that reason we regard  $X$  and  $Y$  as subspaces of  $Z_f$ .)

(b) Prove that  $Y$  is a deformation retract of  $Z_f$ .

(c) Let  $A \subset X$  and  $B \subset Y$  be subspaces with  $f(A) \subset B$  and denote by  $g : A \rightarrow B$  the resulting restriction of  $f$ . Show that we may regard  $Z_g$  as a subspace of  $Z_f$  and that we have a long exact sequence

$$\cdots \rightarrow H_k(X, A) \xrightarrow{f_k} H_k(Y, B) \rightarrow H_k(Z_f, X \cup Z_g) \rightarrow H_{k-1}(X, A) \rightarrow \cdots$$

(Hint: Consider the space triple  $(Z_f, X \cup Z_g, Z_g)$ .)

(d) Prove that  $f : (X, A) \rightarrow (Y, B)$  induces an isomorphism on homology precisely when  $(Z_f, X \cup Z_g)$  has zero homology.

### PROBLEM 13.

(a) We begin with a basic result in homological algebra. Let  $\phi : C_\bullet \rightarrow D_\bullet$  be a chain map between exact complexes. Suppose that  $\phi_k$  is an isomorphism if  $k$  belongs to two distinct residue classes modulo three. Prove that  $\phi_k$  is then an isomorphism is for all  $k \in \mathbb{Z}$ .

(b) Let  $\phi : C_\bullet \rightarrow D_\bullet$  be a chain map of complexes and let  $C'_\bullet \subset C_\bullet$  and  $D'_\bullet \subset D_\bullet$  be subcomplexes with  $\phi_k(C'_k) \subset D'_k$  for all  $k \in \mathbb{Z}$ . So  $\phi$  determines chain maps  $\phi' : C'_\bullet \rightarrow D'_\bullet$  and  $\bar{\phi} : C_\bullet/C'_\bullet \rightarrow D_\bullet/D'_\bullet$ . Prove that if two of the three chain maps  $\phi, \phi'$  or  $\bar{\phi}$  give isomorphisms on homology, then so does the third.

(c) Let  $f : (X, Y, Z) \rightarrow (X', Y', Z')$  be a map of space triples. Such a map determines continuous maps of space pairs  $(X, Y) \rightarrow (X', Y')$ ,  $(X, Z) \rightarrow (X', Z')$  and  $(Y, Z) \rightarrow (Y', Z')$ . Prove that if two of these maps give isomorphisms on homology, then the remaining one has that property as well.

PROBLEM 14. This problem is about an algebraic analogue of homotopy. First a definition. Given chain maps  $\phi, \phi' : C_\bullet \rightarrow D_\bullet$ , then a *chain homotopy* from  $\phi$  to  $\phi'$  is a collection of homomorphisms  $(I_k : C_k \rightarrow D_{k+1})_k$  with the property that  $\phi'_k - \phi_k = I_{k-1}d_k + d_{k+1}I_k$  for all  $k$ ;  $\phi$  and  $\phi'$  are then said to be (*chain*) *homotopic*. So with this terminology Lemma 2.16 can be stated as saying that for every space  $X$ ,  $\beta : C_\bullet(X) \rightarrow C_\bullet(X)$  is a chain homotopic to the identity map.

(a) Prove that being homotopic is an equivalence relation on the collection of chain maps from  $C_\bullet$  to  $D_\bullet$ . (An equivalence class will be called a *chain homotopy class*.)

(b) Prove that chain homotopy classes can be composed: if  $\phi, \phi' : C_\bullet \rightarrow D_\bullet$  and  $\psi, \psi' : D_\bullet \rightarrow E_\bullet$  are homotopic chain maps, then so are  $\psi\phi, \psi'\phi' : C_\bullet \rightarrow E_\bullet$ .

(c) Prove that homotopic chain maps  $\phi, \phi' : C_\bullet \rightarrow D_\bullet$  induce the same maps on homology:  $H_k(\phi) = H_k(\phi')$  for all  $k$ .

(d) Prove that if  $f, f' : X \rightarrow Y$  are homotopic, then so are the chain maps  $f_*, f'_* : C_\bullet(X) \rightarrow C_\bullet(Y)$ .

## 3. Some applications

This chapter depends on the preceding chapter only via the fundamental properties 0-IV of homology

We denote by  $\dot{\Delta}^n$  (the ‘boundary’ of  $\Delta^n$ ) the union of the  $(n-1)$ -faces of  $\Delta^n$ . It is easily verified that the pair  $(\Delta^n, \dot{\Delta}^n)$  is homeomorphic to the pair  $(B^n, S^{n-1})$ ,



consisting of the (closed) unit  $n$ -ball  $B^n$  and its boundary  $S^{n-1}$ , the unit  $(n-1)$ -sphere. So the next theorem gives us the homology of  $(B^n, S^{n-1})$ .

**THEOREM 3.1.** *For  $k \neq n$  we have  $H_k(\Delta^n, \dot{\Delta}^n) = 0$  and*

$$\mathbb{Z} \rightarrow H_n(\Delta^n, \dot{\Delta}^n), \quad 1 \mapsto [\langle e_0, \dots, e_n \rangle],$$

*is an isomorphism.*

**PROOF.** With induction on  $n$ . For  $n = 0$  this is the dimension property 2.1. We now assume  $n > 0$  and suppose the theorem proved for smaller values of  $n$ .

Let  $A \subset \dot{\Delta}^n$  be the union of all  $((n-1)$ -dimensional) faces of  $\dot{\Delta}^n$  distinct from  $\Delta^{n-1}$ . We claim that the homology of the pair  $(\Delta^n, A)$  is zero: the fact that both  $A$  and  $\Delta^n$  are star shaped with respect to  $e_n$  implies that  $(\{e_n\}, \{e_n\}) \subset (\Delta^n, A)$  is a homotopy equivalence and hence  $H_k(\Delta^n, A) \cong H_k(\{e_n\}, \{e_n\}) = 0$ . If we feed this into the exact sequence of the triple  $(\Delta^n, \dot{\Delta}^n, A)$ , we find that for all  $k \in \mathbb{Z}$ ,

$$\partial_k : H_k(\Delta^n, \dot{\Delta}^n) \rightarrow H_{k-1}(\dot{\Delta}^n, A)$$

is an isomorphism.

Next we show that the inclusion of pairs  $(\Delta^{n-1}, \dot{\Delta}^{n-1}) \subset (\dot{\Delta}^n, A)$  induces an isomorphism on homology. We cannot immediately invoke the excision property as its hypotheses are not satisfied. We remedy this with the help the subspace  $B \subset A$  consisting of points in  $A$  whose last barycentric coordinate (the coefficient of  $e_n$ ) is  $\geq \frac{1}{2}$ . Since  $B$  is closed and contained in the interior of  $A$  (viewed as a subspace of  $\dot{\Delta}^n$ ), the excision property tells us that the inclusion  $(\dot{\Delta}^n - B, A - B) \subset (\dot{\Delta}^n, A)$  induces an isomorphism on homology. The inclusion  $i : (\Delta^{n-1}, \dot{\Delta}^{n-1}) \subset (\dot{\Delta}^n - B, A - B)$  has as homotopy inverse the projection  $r$  from  $e_n$  onto  $\Delta^{n-1}$ : for  $ri$  is the identity and

$$F : (\dot{\Delta}^n - B) \times [0, 1] \rightarrow \dot{\Delta}^n - B, \quad F(x, t) = (1-t)r(x) + tx$$

is a homotopy from  $ir$  to the identity. So  $i$  induces an isomorphism on the homology groups. This implies that  $(\Delta^{n-1}, \dot{\Delta}^{n-1}) \subset (\dot{\Delta}^n, A)$  induces in every degree an isomorphism on homology.

Combining the two isomorphisms above yields for every  $k$  the isomorphism

$$H_k(\Delta^n, \dot{\Delta}^n) \xrightarrow{\partial_k} H_{k-1}(\dot{\Delta}^n, A) \cong H_{k-1}(\Delta^{n-1}, \dot{\Delta}^{n-1})$$

By our induction hypothesis, the latter is zero unless  $k-1 = n-1$  (or what amounts to the same,  $k = n$ ), in which case it is isomorphic to  $\mathbb{Z}$ . Let us make this last isomorphism explicit: the class in  $H_n(\Delta^n, \dot{\Delta}^n)$  of  $\langle e_0, \dots, e_n \rangle$  maps under  $\delta_n$  to the class of  $d(\langle e_0, \dots, e_n \rangle) = \sum_{i=0}^n (-1)^i \langle e_0, \dots, \hat{e}_i, \dots, e_n \rangle$  in  $H_{n-1}(\dot{\Delta}^n, A)$ . This is just the class of the chain  $(-1)^n \langle e_0, \dots, e_{n-1} \rangle$ . But that chain even defines an element of  $H_{n-1}(\Delta^{n-1}, \dot{\Delta}^{n-1})$ . So the above isomorphism sends  $[\langle e_0, \dots, e_n \rangle]$  to  $(-1)^n [\langle e_0, \dots, e_{n-1} \rangle]$ . The induction assumption asserts that the latter is a generator of  $H_{n-1}(\Delta^{n-1}, \dot{\Delta}^{n-1})$ . So  $[\langle e_0, \dots, e_n \rangle]$  is a generator of  $H_n(\Delta^n, \dot{\Delta}^n)$ .  $\square$

**COROLLARY 3.2.** *We have for  $n > 0$*

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, n; \\ 0 & \text{otherwise.} \end{cases}$$

*In particular, if  $n \neq m$ , then  $S^n$  and  $S^m$  are not homotopy-equivalent.*

PROOF. If we substitute in the exact homology sequence of the pair  $(B^{n+1}, S^n)$  the fact that  $H_k(B^{n+1}) = 0$  for  $k \neq 0$  and  $H_k(B^{n+1}, S^n) = 0$  for  $k \neq n+1$ , we find that both  $\delta_n : H_{n+1}(B^{n+1}, S^n) \rightarrow H_n(S^n)$  and  $H_0(S^n) \rightarrow H_0(B^n)$  are isomorphisms. Theorem 3.1 resp. the dimension property says that either is isomorphic to  $\mathbb{Z}$ .  $\square$

COROLLARY 3.3 (L.E.J. Brouwer 1912). *If  $n \neq m$ , then  $\mathbb{R}^n$  and  $\mathbb{R}^m$  are not homeomorphic to each other.*

PROOF. If they were, then so would be their one point compactifications. But these are homeomorphic to  $S^n$  and  $S^m$  respectively, and hence according to 3.2, then  $n = m$ .  $\square$

COROLLARY 3.4. *The sphere  $S^n$  is not a retract of  $B^{n+1}$ : there is no map  $r : B^{n+1} \rightarrow S^n$  which is the identity on  $S^n$ .*

PROOF. Otherwise the composite

$$H_n(S^n) \rightarrow H_n(B^{n+1}) \xrightarrow{r_n} H_n(S^n)$$

would be the identity. For  $n > 0$ , this contradicts the fact that  $H_n(S^n) \cong \mathbb{Z}$  and  $H_n(B^{n+1}) = 0$ . The case  $n = 0$  is left to you.  $\square$

COROLLARY 3.5 (Brouwer's fixed point theorem 1910-12). *Every map  $f : B^n \rightarrow B^n$  has a fixed point, that is, an  $x \in B^n$  with the property that  $f(x) = x$ .*

PROOF. We suppose there is no fixed point and derive a contradiction. We define a retraction map  $r : B^n \rightarrow S^{n-1}$ , by letting  $r(x)$  be the unique point of intersection of the the ray  $\{x + t(x - f(x))\}_{t \geq 0}$  'pointing away from  $f(x)$ ' with  $S^{n-1}$ . It is not hard to verify that this point of intersection exists and is unique. It equals  $x$  if  $x \in S^{n-1}$ . The continuity of  $r$  can be verified as follows: since  $B^n$  is compact,  $\delta := \inf\{\|f(x) - x\| \mid x \in B^n\} > 0$ . The subset

$$D := \{(x, y, t) \in B^n \times S^{n-1} \times [0, \infty) \mid y = x + t(x - f(x))\}$$

is closed and bounded (for  $(x, y, t) \in D$  implies  $t \leq \frac{2}{\delta}$ ) and hence compact. The projection from  $D$  onto  $B^n$  is continuous and bijective. Since a continuous map from a compact space to a Hausdorff space is closed, it follows that this projection must then be a homeomorphism. The inverse of this homeomorphism has  $r$  as second component, hence  $r$  is continuous. But this contradicts Corollary 3.4.  $\square$

REMARK 3.6. A map  $f : S^n \rightarrow S^n$ ,  $n > 0$ , determines an endomorphism  $f_*$  of  $H_n(S^n) \cong \mathbb{Z}$ . Such an endomorphism is of course multiplication by a fixed integer. We call this integer the *degree* of  $f$ , and write  $\deg(f)$ . It follows from Corollary 2.7 that the degree of  $f$  only depends on the homotopy class of  $f$ . (It can be shown that the converse also holds: two maps  $f, g : S^n \rightarrow S^n$  of the same degree are homotopic.) Theorem 1.7 implies that identity map of  $S^n$  has degree 1, and that for  $f, g : S^n \rightarrow S^n$ ,  $\deg(gf) = \deg(f)\deg(g)$ . So if  $f$  is a homotopy equivalence (in particular, if  $f$  is a homeomorphism), then  $\deg(f)^{-1} = \deg(g)$  is also an integer and hence equal to  $\pm 1$ .

LEMMA 3.7. *Let  $f : S^n \rightarrow S^n$  be a map with  $f(x) \neq -x$ , resp.  $\neq x$  for all  $x \in S^n$ . Then  $f$  is homotopic to the identity map resp. the antipodal map, and hence has degree 1 resp.  $(-1)^{n+1}$ .*

PROOF. The assumption implies that the segment  $[f(x), x]$  resp.  $[f(x), -x]$  does not contain 0. So we may define a homotopy from  $f$  to  $\pm 1$  by

$$F(x, t) = \frac{(1-t)f(x) \pm tx}{\|(1-t)f(x) \pm tx\|}.$$

For the last clause we refer to 3.11 and Exercise 19.  $\square$

COROLLARY 3.8. *For every map  $f : S^{2m} \rightarrow S^{2m}$  there exists an  $x \in S^{2m}$  with  $f(x) \in \{\pm x\}$ .*

PROOF. Otherwise  $f$  would have degree 1 (by the previous lemma) and degree  $(-1)^{2m+1} = -1$ .  $\square$

We shall later find that in the situation of this lemma  $f$  always has a fixed point (so we take the plus sign).

By a *vector field* on  $S^n$  we mean a map  $v : S^n \rightarrow \mathbb{R}^{n+1}$  with the property that  $v(x) \perp x$  for all  $x \in S^n$  (we usually think of  $v(x)$  as a vector with base point  $x$  rather than the origin; this makes it tangent to  $S^n$  at  $x$ ).

COROLLARY 3.9. *Any vector field on an even dimensional sphere  $S^{2m}$  has a zero.*

PROOF. Suppose there exists a vector field  $v : S^{2m} \rightarrow \mathbb{R}^{2m+1}$  without zeroes. Then  $\frac{v}{\|v\|} : S^{2m} \rightarrow S^{2m}$  is a map which does not take any point into itself or its antipode. This contradicts Corollary 3.8.  $\square$

On the other hand, there exists a nowhere zero vector field on any sphere of odd-dimension: view  $S^{2m+1}$  as the unit sphere of  $\mathbb{C}^{m+1}$  and take  $v(z) = \sqrt{-1}z$ .

We next discuss a local notion of degree. Given a space  $X$ , a closed subset  $F$  of  $X$  and a neighborhood  $U$  of  $F$  in  $X$ , then the excision theorem (applied to the open  $A := X - F$  and the closed  $B := X - U$  contained in  $A$ ) asserts that the inclusion  $(U, U - F) \subset (X, X - F)$  induces an isomorphism on homology. So  $H_k(X, X - F)$  only depends on a neighborhood of  $F$  in  $X$ . Of special interest is the case when  $F$  is a singleton  $F = \{p\}$ ; we then call  $H_k(X, X - \{p\})$  the *k-th local homology group* of  $p \in X$ .

It is remarkable that in case the ambient space  $X$  is a finite dimensional real vector space this can be connected to the notion of orientation. We first recall the following result from linear algebra.

LEMMA 3.10. *Let  $V$  be real vector space of finite positive dimension  $n > 0$ . Then the collection  $\mathcal{B}(V) \subset V^n$  of ordered bases of  $V$  form an open subset of  $V^n$  with two connected components. Two bases  $(v_1, \dots, v_n)$  and  $(v'_1, \dots, v'_n)$  of  $V$  belong to the same component precisely when the matrix  $(m_i^j)$  that expresses the former in the latter  $v'_i = \sum_j m_i^j v_j$  has positive determinant.*

PROOF. If we fix a basis  $(v_1, \dots, v_n)$  of  $V$ , then the map  $\text{GL}(n, \mathbb{R}) \rightarrow \mathcal{B}(V)$ ,  $(m_i^j) \mapsto (\sum_j m_i^j v_j)_i$ , is a bijection. This bijection is continuous and open and hence a homeomorphism. That makes the lemma equivalent to the assertion that  $\text{GL}(n, \mathbb{R})$  has two connected components which are separated by the sign of the determinant  $(m_i^j) \in \text{GL}(n, \mathbb{R}) \mapsto \det(m_i^j)/|\det(m_i^j)| \in \{1, -1\}$ . We suppose this fact known (this is essentially equivalent to the fact that the group  $\text{SL}(n, \mathbb{R})$  is connected).  $\square$

Such a connected component is called an *orientation* of  $V$  and if one of these has been singled out, then we say that  $V$  is *oriented*. The following lemma shows that an orientation of  $V$  can be given in purely topological terms as the choice of a generator of  $H_n(V, V - \{0\})$ .

LEMMA 3.11. *Let  $V$  be a real vector space of finite positive dimension  $n > 0$ . Then  $H_k(V, V - \{0\}) = 0$  when  $k \neq n$ . Given a basis  $\mathbf{v} = (v_1, \dots, v_n)$  of  $V$ , put  $v_0 := -\sum_{i=1}^n v_i$ . Then  $\langle v_0, \dots, v_n \rangle \in Z_n(V, V - \{0\})$  and*

$$\mathbb{Z} \rightarrow H_n(V, V - \{0\}), \quad 1 \mapsto [\langle v_0, \dots, v_n \rangle]$$

*is an isomorphism. So the image of  $1 \in \mathbb{Z}$  is a generator  $u_{\mathbf{v}}$  of  $H_n(V, V - \{0\})$ . This generator only depends on the orientation of  $V$  defined by  $\mathbf{v}$  and opposite orientations define opposite generators.*

PROOF. We first observe that  $\langle v_0, \dots, v_n \rangle$  maps the barycenter  $z$  of  $\Delta^n$  to 0 and that 0 is in the interior of the image of  $\langle v_0, \dots, v_n \rangle$ . According to the preceding,  $H_k(\langle v_0, \dots, v_n \rangle)$  is then an isomorphism of  $H_k(\Delta^n, \Delta^n - \{z\})$  onto  $H_k(V, V - \{0\})$ . The pair  $(\Delta^n - \{z\}, \dot{\Delta}^n)$  has no homology because the inclusion  $\dot{\Delta}^n \subset \Delta^n - \{z\}$  is a homotopy equivalence. The exactness of the homology sequence of the triple  $(\Delta^n, \Delta^n - \{z\}, \dot{\Delta}^n)$  then shows that  $H_k(\Delta^n, \dot{\Delta}^n) \rightarrow H_k(\Delta^n, \Delta^n - \{z\})$  is an isomorphism. The first clause of the lemma now follows 3.1.

For the proof of its last assertion, we observe that if two bases of  $V$  lie in the same connected component of  $\mathcal{B}(V)$ , then the resulting maps of pairs  $(\Delta^n, \dot{\Delta}^n) \rightarrow (V, V - \{0\})$  are homotopic and hence yield the same generator of  $H_n(V, V - \{0\})$ . A transformation which changes orientation can be represented by the map  $s$  which exchanges the first two basis vectors of the basis  $v_1, \dots, v_n$  ( $n \geq 2$ ) resp. is  $-1_V$  ( $n = 1$ ). So  $s$  induces in the singular simplex  $\langle v_0, \dots, v_n \rangle$  the exchange of  $v_1$  en  $v_2$  resp.  $v_0$  en  $v_1$ . It remains to check that in either case this exchange of vertices of  $\Delta^n$  acts in  $H_n(V, V - \{0\})$  as  $-1$ . Or equivalently, that the corresponding exchange of vertices of  $\Delta^n$  has the same effect on  $H_n(\Delta^n, \dot{\Delta}^n)$ . We do this for  $n \geq 2$  only (the case  $n = 1$  being easier). If we work out the boundary of the singular  $(n+1)$ -simplex  $\langle e_0, e_2, e_1, e_2, e_3, \dots, e_n \rangle : \Delta^{n+1} \rightarrow \Delta^n$  modulo  $C_n(\dot{\Delta}^n)$ , then only terms in which  $e_2$  appears only once remain (otherwise the term in question lands in  $\dot{\Delta}^n$ ), in other words, we get

$$\begin{aligned} d(\langle e_0, e_2, e_1, e_2, e_3, \dots, e_n \rangle) \\ \equiv -\langle e_0, e_1, e_2, e_3, \dots, e_n \rangle - \langle e_0, e_2, e_1, e_3, \dots, e_n \rangle \pmod{C_n(\dot{\Delta}^n)}, \end{aligned}$$

It follows that  $\langle e_0, e_2, e_1, e_3, \dots, e_n \rangle$  and  $\langle e_0, e_1, e_2, e_3, \dots, e_n \rangle$  define opposite elements of the infinite cyclic group  $H_n(\Delta^n, \dot{\Delta}^n)$ .  $\square$

If we let  $p \in V$  vary, then the groups  $H_n(V, V - \{p\})$  can be identified with each other in a natural manner, for instance by a translation, but it can also be done by using the topology of  $V$  only: given  $p_1, p_2 \in V$ , then choose a compact subset  $D \subset V$  which contains either and for which the inclusions  $V - D \subset V - \{p_i\}$  yield isomorphisms on homology (the simplest example is perhaps  $D = [p_1, p_2]$ ). Then the inclusions  $(V, V - D) \subset (V, V - \{p_i\})$  also induce isomorphisms on homology. The isomorphism  $H_n(V, V - \{p_1\}) \cong H_n(V, V - \{p_2\})$  thus obtained is independent of the choice of  $D$  because we may choose  $D$  as large as we wish: for another choice  $D'$ , choose a compact ball like  $D''$  containing  $D \cup D'$ ; then the composite

$(V, V - D'') \subset (V, V - D) \subset (V, V - \{p_i\})$  induces an isomorphism on homology and likewise when  $D$  is replaced by  $D'$ .

Let  $V$  and  $V'$  be real oriented vector spaces of the same dimension  $n > 0$ ,  $U \subset V$  open and  $f : U \rightarrow V'$  a map. Suppose  $p \in U$  such that  $p$  is an isolated point of  $f^{-1}f(p)$ . This means that there is a neighborhood  $U_p \ni p$  such that  $f(U_p - \{p\}) \subset W - \{f(p)\}$ . This means that  $p$  is the only solution to the ‘equation’  $f(x) = f(p)$  that is inside  $U_p$ . We attach to this solution a multiplicity as follows. Notice that  $f$  defines a homomorphism

$$H_n(U_p, U_p - \{p\}) \rightarrow H_n(V', V' - \{f(p)\}).$$

The given orientations identify domain and range with  $\mathbb{Z}$  so that this homomorphism is then given as multiplication by an integer  $\lambda \in \mathbb{Z}$ . If we replace  $U_p$  by a smaller neighborhood  $U'_p \ni p$ , then

$$H_n(U'_p, U'_p - \{p\}) \cong H_n(U_p, U_p - \{p\}),$$

and so this will not alter  $\lambda$ . Hence  $\lambda$  only depends of the restriction of  $f$  to an arbitrary small neighborhood of  $p$  in  $U$ . This number is the multiplicity we alluded to; we call it however the *local degree of  $f$  at  $p$* , and denote it  $\deg_p(f)$ . Let us observe that it is multiplicative with respect to composition. In particular, if  $f$  maps a neighborhood  $U_p$  of  $p$  homeomorphically onto a neighborhood of  $f(p)$  with inverse  $g$ , then  $\deg_{f(p)}(g) \cdot \deg_p(f) = \deg_p(gf) = 1$ , so that we must have  $\deg_p(f) = \pm 1$ . Although it is reasonable to interpret the local degree as the multiplicity of a solution, be aware of the possibility that it may take the value zero or can be negative.

**EXAMPLE 3.12.** Let  $f$  be a  $\mathbb{R}$ -valued function defined on an open neighborhood of  $p \in \mathbb{R}$ . If  $f$  is strictly monotone increasing resp. decreasing at  $p$ , then the local degree of  $f$  at  $p$  is 1 (resp.  $-1$ ).

The apparent topological nature of the notion of orientation leads to the following setting. Let  $X$  be a Hausdorff space with the property that every point has a neighborhood homeomorphic to an open subset of  $\mathbb{R}^n$ ; we call such a space a  *$n$ -manifold*. The dimension  $n$  is topologically recognized as the sole degree for which the local homology of  $X$  at  $p$  is nonzero. This nonzero group  $H_n(X, X - \{p\})$  is infinite cyclic and has therefore two generators. The choice of a generator will be called an *orientation of  $X$  at  $p$* . The preceding discussion shows that this then also determines an orientation of  $X$  at the points of a ball like neighborhood  $U_p$  of  $p$ . This suggests the following definition.

**DEFINITION 3.13.** An *orientation*  $\mu$  of the  $n$ -manifold  $X$  assigns to every  $p \in X$  a generator  $\mu_p$  of  $H_n(X, X - \{p\})$  in a manner that is continuous in the sense that we can cover  $X$  by ball like open subsets  $U$ , such that for every pair  $p, q \in U$ , the composite isomorphism  $H_n(X, X - \{p\}) \cong H_n(X, X - U) \cong H_n(X, X - \{q\})$  maps  $\mu_p$  to  $\mu_q$ .

An orientation of an  $n$ -manifold need not exist (an example is the Möbius band). And if an orientation  $\mu$  is given, then  $-\mu$  is also one, called, of course, the *opposite orientation*. Given a map  $f : X \rightarrow Y$  between oriented  $n$ -manifolds, and  $p \in X$  such that  $p$  is isolated in  $f^{-1}f(p)$ , then we have a notion of local degree  $\deg_p(f)$  just as before.

EXAMPLE 3.14. The  $n$ -sphere  $S^n$  is an  $n$ -manifold. An orientation of it for  $n > 0$  can be obtained as follows. We claim that for every  $p$ , the map  $H^n(S^n) \rightarrow H^n(S^n, S^n - \{p\})$  is an isomorphism. This follows from the fact that  $S^n - \{p\}$  is contractible and inspection of the exact sequence of the pair  $(S^n, S^n - \{p\})$ . Choose a generator  $\mu \in H^n(S^n)$ . Now if we let  $\mu_p$  be the image of  $\mu$  under the above isomorphism, then  $p \mapsto \mu_p$  is easily shown to define an orientation.

PROPOSITION 3.15. *Let  $f : S^n \rightarrow S^n$  ( $n > 0$ ) be a map. If  $q \in S^n$  is such that  $f^{-1}(q)$  is finite, then  $\deg(f) = \sum_{p \in f^{-1}(q)} \deg_p(f)$ .*

PROOF. For every  $p \in f^{-1}(q)$  we choose an open neighborhood  $U_p$  of  $p$  in  $S^n$  such that  $U_p \cap U_{p'} = \emptyset$  in case  $p \neq p'$  and we put  $U = \cup_p U_p$ . Consider the commutative diagram

$$\begin{array}{ccc}
H_n(S^n) & \xrightarrow{H_n(f)} & H_n(S^n) \\
\downarrow & & \downarrow \cong \\
H_n(S^n, S^n - f^{-1}(q)) & \xrightarrow{H_n(f)} & H_n(S^n, S^n - \{q\}) \\
\cong \uparrow & & \\
H_n(U, U - f^{-1}(q)) & & \\
\cong \uparrow & & \\
\bigoplus_{p \in f^{-1}(q)} H_n(U_p, U_p - \{p\}) & & .
\end{array}$$

If we follow the left column from top to bottom, then we find for every  $p \in f^{-1}(q)$  a homomorphism

$$H_n(S^n) \rightarrow H_n(U_p, U_p - \{p\}) \cong H_n(S^n, S^n - \{p\}).$$

This is in fact the isomorphism that takes  $\mu$  to  $\mu_p$ . By definition the top arrow sends  $\mu$  to  $\deg(f)\mu$  and the composite

$$\begin{aligned}
H_n(S^n, S^n - \{p\}) \cong H_n(U_p, U_p - \{p\}) &\rightarrow H_n(U, U - f^{-1}(q)) \rightarrow \\
&\rightarrow H_n(S^n, S^n - f^{-1}(q)) \rightarrow H_n(S^n, S^n - \{q\})
\end{aligned}$$

sends  $\mu_p$  to  $\deg_p(f)\mu_q$ . The proposition follows.  $\square$

This proposition may help us computing the degree of  $f$ . If for example  $f$  is a local homeomorphism in every point of  $f^{-1}(q)$ , then the local degree of  $f$  in each such point is  $\pm 1$  and the degree of  $f$  is the sum of these. In particular,  $\deg(f)$  is then bounded by the cardinality of  $f^{-1}(q)$  and the difference is even.

Next we show how to compute the degree of a map  $f : S^1 \rightarrow S^1$ . If we regard  $S^1$  as the unit circle of  $\mathbb{C}$ , then  $f$  may be given by a continuous function  $\phi : [0, 2\pi] \rightarrow \mathbb{R}$  such that  $f(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1}\phi(\theta)}$ . It is clear that  $\phi(2\pi) - \phi(0) = 2k\pi$  for some  $k \in \mathbb{Z}$ .

PROPOSITION 3.16. *The map  $f$  is homotopy equivalent to  $\mu_k : z \in S^1 \mapsto z^k \in S^1$  and  $\deg(f) = k$ .*

PROOF. Composing  $f$  with rotation over  $-\phi(0)$  neither affects  $k$  nor the homotopy class of  $f$  and so we may assume without loss of generality that  $\phi(0) = 0$ .

Define  $\phi_t : ([0, 2\pi], \{0, 2\pi\}) \rightarrow (\mathbb{R}, \{0, 2\pi k\})$  by  $\phi_t(\theta) = (1-t)\phi(\theta) + tk\theta$ . This determines a homotopy from  $f$  to  $\mu_k$ . So  $\deg(f) = \deg(\mu_k)$ . For  $k \neq 0$ ,  $\mu_k$  maps any small enough open angle of  $S^1$  homeomorphically onto an open angle and this map is orientation preserving or reversing according to the sign of  $k$ . Since a fiber of  $\mu_k$  has exactly  $k$  points, it then follows from Proposition 3.15 that  $\mu_k$  has degree  $k$ . This is evidently also true when  $k = 0$ .  $\square$

Here is another geometric application of Proposition 3.15. It essentially states that under suitable assumptions the number of solutions of  $n$  equations with  $n$  unknowns,  $f_i(x_1, \dots, x_n) = q_i, i = 1, \dots, n$ , does not change under small perturbation of the constants  $q_i$ , as long as we count every solution with its multiplicity. In complex function theory this is known as Rouché's principle. (But remember that such a multiplicity need not be positive.)

**PROPOSITION 3.17.** *Let  $f : X \rightarrow Y$  be a closed map between oriented  $n$ -manifolds whose fibers are finite. Then  $\sum_{p \in f^{-1}(q)} \deg_p(f)$  is locally constant in  $q \in Y$ .*

The proof uses the following property of  $n$ -manifolds:

**LEMMA 3.18.** *Every point  $p$  of an  $n$ -manifold  $X$  has an open neighborhood  $U$  with the property that the quotient space  $X/(X - U)$  (obtained from  $X$  by collapsing  $X - U$  to a point) is homeomorphic to  $S^n$ .*

**PROOF.** Let  $\kappa : U_p \rightarrow \mathbb{R}^n$  be a homeomorphism of an open  $U_p \ni p$  in  $X$  onto an open part of  $\mathbb{R}^n$  with  $\kappa(p) = 0$ . After multiplication of  $\kappa$  by a scalar, we may assume that  $\kappa(U_p)$  contains the closed unit ball  $B^n$  of  $\mathbb{R}^n$ . Then  $U := \kappa^{-1}(B^n - S^{n-1})$  has the desired property, for  $\kappa^{-1}$  then defines a map  $(B^n, S^{n-1}) \rightarrow (X, X - U)$  and such a map induces a (continuous) map  $B^n/S^{n-1} \rightarrow X/(X - U)$ . This map is also bijection from a compact space to a Hausdorff space, hence is a homeomorphism. The lemma now follows from the observation that stereographic projection identifies  $B^n/S^{n-1}$  with  $S^n$ .  $\square$

**PROOF OF PROPOSITION 3.17.** Let  $q \in Y$ . We first treat the special (yet essential) case when  $f^{-1}(q)$  consists of a single point  $p$ . Choose  $U$  as in Lemma 3.18. Then  $X - U$  is closed in  $X$  and so  $f(X - U)$  is closed in  $Y$ . Hence  $V' := Y - f(X - U)$  is an open neighborhood of  $q$  in  $Y$  with the property that  $f(X - U) = Y - V'$ . We apply Lemma 3.18 once again to  $V'$  and find an open neighborhood  $V \ni q$  in  $V'$  with the property that  $Y/(Y - V)$  is homeomorphic to  $S^n$ . Thus  $f$  yields a map of pairs  $(X, X - U) \rightarrow (Y, Y - V)$ . This induces a map  $S^n \cong X/(X - U) \rightarrow Y/(Y - V) \cong S^n$  to which Proposition 3.15 applies: we find that  $y \in V \mapsto \sum_{x \in f^{-1}(y)} \deg_x(f)$  is constant.

The general case involves a bit of technique, but hardly a new idea. Choose for every  $p \in f^{-1}(q)$  an open  $U_p \ni p$  in  $X$  such that  $U_p \cap U_{p'} = \emptyset$  if  $p \neq p'$ . Since  $f$  is closed,  $f(X - \cup_{p \in f^{-1}(q)} U_p)$  is closed in  $Y$  and so  $V := Y - f(X - \cup_p U_p)$  is open in  $Y$ . We thus have found a neighborhood  $V$  of  $q$  with  $f^{-1}V \subset \cup_p U_p$ . Put  $U'_p := U_p \cap f^{-1}V$ . A simple exercise in set theory shows that the restriction  $f_p : U'_p \rightarrow V$  of  $f$  is still closed: any closed subset of  $U'_p$  is of the form  $G \cap U'_p$  with  $G$  closed in  $X$ . Since the difference  $G - \cup_{p' \neq p} U'_{p'}$  is also closed in  $X$  and meets  $U_p$  in  $G \cap U_p$ , we may assume that  $G$  does not meet  $U_{p'}$  when  $p' \neq p$ . This implies that  $f_p(G \cap U'_p) = f(G) \cap V$  and the latter is closed in  $V$ . Hence  $f_p$  is of the type just treated (with  $f_p^{-1}(q) = p$ ) and so  $y \in V \mapsto \sum_{x \in f_p^{-1}(y)} \deg_x(f)$  is constant on a

neighborhood of  $q$ . Summing over  $p \in f^{-1}(q)$  yields that this is then also true for  $y \mapsto \sum_{x \in f^{-1}(y)} \deg_x(f)$ .  $\square$

If  $Y$  is connected, then Proposition 3.17 says that  $\sum_{p \in f^{-1}(q)} \deg_p(f)$  is independent of  $q$ ; this number is also called the *degree* of  $f$ .

**EXAMPLE 3.19.** Consider for  $n > 0$ , a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  of the form  $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ . This function has finite fibers and is closed (this follows from the fact that  $\lim_{x \rightarrow \pm\infty} |f(x)| = \infty$ , check this). Its degree is zero when  $n$  is even, for then  $f$  is bounded from below and this implies that for sufficiently small  $q \in \mathbb{R}$ , the fiber  $f^{-1}(q)$  is empty. However for  $n$  odd the degree of  $f$  is 1: for  $q$  large enough there is just one solution  $x = x(q)$  for  $f(x) = q$ , and this solution grows then monotone with  $q$ . The local degree of  $f$  at  $x(q)$  is then 1 and so that is also the degree of  $f$ .

**REMARK 3.20.** The following geometric description of the notion of a singular  $k$ -cycle, although not used in what follows, may have more intuitive appeal than the original definition.

Let  $X$  be a space and  $\Sigma = \{\sigma_i : \Delta^k \rightarrow X\}_{i \in I}$  a finite collection singular  $k$ -simplices of  $X$  (so  $I$  is finite) that are pairwise distinct. We give  $I$  the discrete topology and define a quotient space  $|\Sigma|$  on  $\Delta^k \times I$  as follows: for every  $i \in I$ , we consider the terms that appear in  $d\sigma_i$ : for  $j = 0, \dots, k$ , the singular  $(k-1)$ -simplex  $\sigma_i^j := \sigma_i \langle e_0, \dots, \widehat{e}_j, \dots, e_k \rangle : \Delta^{k-1} \rightarrow X$ . We use the same pair of indices  $(i, j)$  to identify the faces of  $\Delta^k \times I$  and we identify two such (in an affine-linear manner) precisely when the corresponding singular  $(k-1)$ -simplices coincide. The resulting quotient  $|\Sigma|$  is a Hausdorff space and comes with a map  $S : |\Sigma| \rightarrow X$  such that every  $\sigma_i$  naturally lifts to  $\tilde{\sigma}_i : \Delta^k \rightarrow X$ . The images of the relatively open facets of  $\Delta^k$  under the  $\tilde{\sigma}_i$ 's decompose  $\tilde{\Sigma}$  (but we are not claiming that these are homeomorphic images). If  $\Sigma$  contains the support of a  $k$ -chain  $c \in C_k(X)$ :  $c = \sum_{i \in I} n_i \sigma_i$ , then  $\tilde{c} := \sum_i n_i \tilde{\sigma}_i$  is a lift of  $c$  in  $C_k(|\Sigma|)$ . If  $c$  is cycle on  $X$ , then  $\tilde{c}$  is one on  $|\Sigma|$ . Since  $|\Sigma|$  is in general a simpler kind of space than  $X$ , this makes the notion of  $k$ -cycle perhaps more concrete.

Actually, in the preceding construction, it was not really necessary to take  $I$  finite. In fact, we could have done this for the collection of *all* singular simplices. The resulting space is then infinite dimensional, yet comes with a map to  $X$  which induces an isomorphism on homology.

**PROBLEM 15.** Let  $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$  be as above and let  $p \in \mathbb{R}$ . Let  $k$  the first positive integer for which  $f^{(k)}(p) \neq 0$ . Prove that the local degree of  $f$  in  $p$  is zero for  $k$  even and the sign of  $f^{(k)}(p)$  for  $k$  odd.

**PROBLEM 16.** Compute the homology of the space obtained by removing  $m > 0$  distinct points from  $\mathbb{R}^n$  ( $n > 0$ ). Conclude that two such spaces are not homotopy equivalent unless the associated pairs  $(n, m)$  are equal.

**PROBLEM 17.** For  $m < n$  we consider  $S^m$  as a subspace of  $S^n$  defined by putting the last  $n - m$  coordinates equal to zero. Prove that  $S^m$  is not a retract of  $S^n$ .

**PROBLEM 18.** Let  $X$  be a space and  $r : X \rightarrow \{o\}$  the unique map to a singleton. The  $k$ -th reduced homology group of  $X$ , denoted  $\tilde{H}_k(X)$ , is the kernel of  $H_k(r) : H_k(X) \rightarrow H_k(\{o\})$ .



- (a) Prove that for every singleton subset  $P \subset X$  and every  $k$  the composite  $\tilde{H}_k(X) \rightarrow H_k(X) \rightarrow H_k(X, P)$  is an isomorphism.
- (b) Prove that for nonempty  $X$ , both maps are isomorphisms if  $k > 0$ , whereas for  $k = 0$  the natural map  $\tilde{H}_0(X) \oplus \mathbb{Z} \cong \tilde{H}_0(X) \oplus H_0(P) \rightarrow H_0(X)$  is an isomorphism.
- (c) Prove that for a map  $f : X \rightarrow Y$ ,  $H_k(f)$  maps  $\tilde{H}_k(X)$  to  $\tilde{H}_k(Y)$ .

PROBLEM 19. Let  $n > 0$  and let  $f$  be an orthogonal transformation of  $\mathbb{R}^n$ . We regard  $f$  as a map from  $S^{n-1}$  to itself. Prove that  $\deg(f) = \det(f)$ . (Hint: use that  $SO_n$  is arcwise connected.)

PROBLEM 20. At every moment there is a point on earth where the wind speed is zero. Explain why.

PROBLEM 21. The degree of differentiable maps.

- (a) Let  $f$  be an invertible linear transformation of  $\mathbb{R}^n$ . Prove that  $\deg_0(f)$  equals the sign of  $\det(f)$ .
- (b) Prove that the local degree at 0 of an invertible complex-linear transformation  $\text{van } \mathbb{C}^m$  is 1.
- (c) Let  $U \subset \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}^n$  a  $C^1$ -map and  $p \in U$  such that the Jacobian matrix  $df(p) = (\frac{\partial f^j}{\partial x^i}(p))_{i,j}$  is invertible. Prove that  $\deg_p(f)$  is the sign of  $\det(df(p))$ .
- (d) Let  $U \subset \mathbb{C}^m$  be open,  $f : U \rightarrow \mathbb{C}^m$  a holomorphic map and  $p \in U$  such that the Jacobian matrix  $df(p) = (\frac{\partial f^j}{\partial z^i}(p))_{i,j}$  is invertible. Prove that  $\deg_p(f) = 1$ .

PROBLEM 22. Let  $f(z) = z^k + a_1 z^{k-1} + \dots + a_k$  ( $k > 0$ ) be a complex polynomial. We regard  $f$  as a map  $\mathbb{C} \rightarrow \mathbb{C}$  and extend  $f$  to the Riemann sphere  $\hat{\mathbb{C}}$  (=one point compactification of  $\mathbb{C}$ ),  $\hat{f} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ , by putting  $\hat{f}(\infty) = \infty$ .

- (a) Prove that  $\hat{f}$  is continuous.
- (b) Let  $f_k(z) := z^k$ . Prove that  $\hat{f}$  and  $\hat{f}_k$  are homotopic.
- (c) Prove that  $\hat{f}_k$  has degree  $k$ .
- (d) Prove that  $f$  is onto. In particular, recover the ‘main theorem of algebra’:  $f$  has a zero.

PROBLEM 23. Let  $f$  be a nonconstant holomorphic function defined on a neighborhood  $p \in \mathbb{C}$ . Prove that its local degree at  $p$  is defined and equal to the smallest  $k > 0$  for which  $f^{(k)}(p) \neq 0$ .

## 4. Cell complexes

*In this chapter all spaces are assumed to have the Hausdorff property.*

Many spaces and space pairs can be endowed with a structure which allows us to compute their homology. This structure, that of a cell complex, is the main topic of this chapter.

DEFINITION 4.1. Let  $X$  be a Hausdorff space. A subspace  $e \subset X$  is called an *n-cel* if there exists a map  $j : B^n \rightarrow X$  which maps  $(B^n)^\circ$  homeomorphically onto  $e$ . Such a map  $j$  is called a *characteristic map* for  $e$ .

We shall denote the boundary of a cell  $e$  by  $\dot{e}$ , so  $\dot{e} := \bar{e} - e$ .

LEMMA 4.2. *In this situation  $j(B^n) = \bar{e}$ ,  $j(S^{n-1}) = \dot{e}$  and  $j$  induces a homeomorphism  $B^n/S^{n-1} \cong \bar{e}/\dot{e}$  (recall that stereographic projection defines a homeomorphism of  $B^n/S^{n-1}$  onto  $S^n$ ).*

PROOF. We first note that the map  $j$  is closed as its domain is compact and its range Hausdorff. In particular,  $j(B^n)$  is closed in  $X$  so that  $j(B^n) \supset \bar{e}$ . Since  $j$  is continuous we also have  $j(B^n) \subset \bar{e}$  and so we have  $j(B^n) = \bar{e}$ .

The assertion that  $j(S^{n-1}) = \dot{e}$  means that  $j(S^{n-1}) \cap e = \emptyset$ , in other words, that for every  $p \in (B^n)^\circ$ , we have  $j(p) \notin j(S^{n-1})$ . Let  $U_p$  be an open ball centered at  $p$  in  $(B^n)^\circ$  whose closure does not meet  $S^{n-1}$ . Since  $j$  maps  $(B^n)^\circ$  homeomorphically onto  $e$ ,  $j(U_p)$  is open in  $e$ , hence of the form  $V_p \cap e$  with  $V_p$  open in  $X$ . Now  $j^{-1}V_p$  is open in  $B^n$  and equal to the union of  $U_p$  and possibly a subset of  $S^{n-1}$ . But the closure of  $U_p$  is disjoint with  $S^{n-1}$  and so this can only be if that subset of  $S^{n-1}$  is empty. In particular,  $j(p) \notin j(S^{n-1})$ .

Now consider the bijection  $\bar{j} : B^n/S^{n-1} \rightarrow \bar{e}/\dot{e}$  induced by  $j$ . On either side we have the quotient topology and this makes  $\bar{j}$  automatically continuous (an open subset of  $\bar{e}/\dot{e}$  is the image of an open subset of  $\bar{e}$  that is either contains  $\dot{e}$  or is contained in  $e$  and likewise for  $B^n/S^{n-1}$ ). Since  $B^n/S^{n-1}$  is compact, it suffices to see that  $\bar{e}/\dot{e}$  is Hausdorff. Since  $e$  is open in the Hausdorff space  $\bar{e}$ , we only need to check that any  $p \in e$  and  $\dot{e}$  have disjoint neighborhoods. This follows from the Hausdorff property of  $X$  and the compactness of  $\dot{e}$ .  $\square$

The preceding lemma shows that any characteristic map for  $e$  restricts to a continuous map  $f : S^{n-1} \rightarrow \dot{e} \subset X - e$ . Of special interest is the case when  $e$  is open in  $X$  (so that  $X - e$  is closed in  $X$ ), such a restriction is sometimes called an *attaching map* for  $e$ . We then may recover  $X$  from  $X - e$  and the attaching map:

LEMMA 4.3. *If in the preceding lemma,  $e$  is open in  $X$  and  $Y := X - e$ , then the obvious map  $\bar{e}/\dot{e} \rightarrow X/Y$  is a homeomorphism and  $X$  has the quotient topology with respect to the evident surjection  $B^n \sqcup Y$  to  $X$  (in other words,  $X$  can be identified with the quotient of  $B^n \cup_f Y$  of  $B^n \sqcup Y$  obtained by identifying  $p \in S^{n-1}$  with  $f(p) \in Y$ ).*

PROOF. The proof of the first assertion is easy and is left to you. For the second, it suffices to show that a subset of  $X$  is closed if and only if its preimage in  $B^n \sqcup Y$  is. The map  $B^n \sqcup Y \rightarrow X$  is evidently continuous and so this comes down to verifying that it is closed as well. This is clearly the case: its restriction to  $Y$  is closed since it simply the inclusion of the closed subset  $Y$  of  $X$  in  $X$  and its restriction to  $B^n$  is closed since  $B^n$  is compact and  $X$  Hausdorff.  $\square$

We then say that  $B^n \cup_f Y$  is obtained from  $Y$  by attaching an  $n$ -cell.

LEMMA 4.4. *A characteristic map  $j : B^n \rightarrow X$  defines an isomorphism*

$$H_k(B^n, S^{n-1}) \xrightarrow{\cong} H_k(\bar{e}, \dot{e})$$

for all  $k \in \mathbb{Z}$ . In particular,  $H_k(\bar{e}, \dot{e}) = 0$  for  $k \neq n$  and  $H_n(\bar{e}, \dot{e})$  is isomorphic to  $\mathbb{Z}$ . If  $e$  is open in  $X$ , then the inclusion also defines isomorphisms  $H_k(\bar{e}, \dot{e}) \rightarrow H_k(X, X - e)$  ( $k \in \mathbb{Z}$ ).

PROOF. Let  $D \subset B^n$  be the thick sphere defined by  $\frac{1}{2} \leq \|x\| \leq 1$ . With the help of the homotopy  $F : (B^n \times [0, 1], S^{n-1} \times [0, 1]) \rightarrow (B^n, D)$ ,

$$F(x, t) = \begin{cases} (1+t)x & \text{if } \|(1+t)x\| \leq 1; \\ \frac{x}{\|x\|} & \text{otherwise,} \end{cases}$$

it follows that  $(B^n, S^{n-1}) \subset (B^n, D)$  is a homotopy equivalence (with  $F_1 : (B^n, D) \rightarrow (B^n, S^{n-1})$  as homotopy inverse). This homotopy composed with  $j$  shows that

$$(\bar{e}, j(D)) \subset (\bar{e}, \dot{e})$$

is a homotopy equivalence as well. So it suffices to prove that the map of pairs  $(B^n, D) \rightarrow (\bar{e}, j(D))$  induces isomorphisms on homology. But this follows from the fact that after excision of  $S^{n-1}$  in the domain and  $\dot{e}$  in the range we get a homeomorphism of pairs. The proof of the second assertion is similar.  $\square$

The choice of generator of the infinite cyclic group  $H_n(\bar{e}, \dot{e})$  is called an *orientation* of  $e$ ; we will usually indicate the presence of an orientation on a cell  $e$  by the use boldface:  $\mathbf{e}$ . We can also express this in terms of reduced homology as defined in Problem 18: if  $P_n \subset S^n \cong B^n/S^{n-1}$  is the image of  $S^{n-1}$  (so a singleton), then the maps in

$$H_n(B^n, S^{n-1}) \rightarrow H_n(S^n, P_n) \leftarrow \tilde{H}_n(S^n)$$

are isomorphisms. Similarly we can identify  $H_n(\bar{e}, \dot{e})$  with  $\tilde{H}_n(\bar{e}/\dot{e})$ , which in case  $e$  is open in  $X$ , can also be identified with  $\tilde{H}_n(X/(X - e))$ . So in that last case, an orientation of  $e$  amounts the choice of a generator of  $\tilde{H}_n(X/(X - e))$ .

**COROLLARY 4.5.** *Let  $X$  be a space and  $Y \subset X$  a closed subspace such that  $e := X - Y$  is an  $n$ -cel. Let  $f : S^{n-1} \rightarrow Y$  be an attaching map for  $e$ . Then for every subspace  $A \subset Y$ , the inclusion  $(Y, A) \subset (X, A)$  induces an isomorphism on homology in degree  $\neq n-1, n$  and for the remaining homology groups we have an exact sequence*

$$0 \rightarrow H_n(Y, A) \rightarrow H_n(X, A) \rightarrow \tilde{H}_{n-1}(S^{n-1}) \xrightarrow{H_{n-1}(f)} H_{n-1}(Y, A) \rightarrow H_{n-1}(X, A) \rightarrow 0.$$

**PROOF.** Consider for any  $k \in \mathbb{Z}$  the commutative square

$$\begin{array}{ccc} H_k(X, Y) & \xrightarrow{\partial_k} & H_{k-1}(Y, A) \\ H_k(j) \uparrow & & \uparrow H_{k-1}(f) \\ H_k(B^n, S^{n-1}) & \xrightarrow{\partial_k} & H_{k-1}(S^{n-1}, P_{n-1}). \end{array}$$

Now use the fact that  $H_k(j)$  and the boundary operator in the bottom are isomorphisms and that  $H_{k-1}(S^{n-1}, P_{n-1})$  may be identified with  $\tilde{H}_{k-1}(S^{n-1})$ .  $\square$

As we have now a reasonable picture of how the attachment of a cell affects homology, we take a closer look at the spaces obtained by repeating this procedure.

**DEFINITION 4.6.** Let  $X$  be a Hausdorff space and  $A$  a subspace. A *CW-structure* on  $(X, A)$  is a decomposition of  $X - A$  into cells such that

- (C) (*Closure finite*) for every cell  $e$ ,  $\bar{e}$  meets only cells of dimension  $< \dim(e)$  and these are finite in number,
- (W) (*Weak topology*) a subset of  $X$  is closed precisely when its intersection with  $A$  and with the closure of every cell is so.

The pair  $(X, A)$  endowed with such a structure is called a *relative CW complex* (of dimension  $\leq n$  if there are no cells of dimension  $> n$ ), where we may omit the adjective ‘relative’ when  $A = \emptyset$ . We say that the CW complex is oriented in case all its cells have been oriented.

**REMARK 4.7.** If there are only finitely many cells, then (W) is automatically satisfied: if  $G \subset X$  meets  $A$  and every cell closure in a closed set, then  $G$  is a finite union of closed sets and hence closed.

A simple example satisfying (C) but not (W) is when  $X = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\}$ , viewed as a subspace of  $\mathbb{R}$  (so Hausdorff),  $A = \{0\}$ , with the singletons  $\{\frac{1}{n}\}$  as the

proposed 0-cells: then  $X - A = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\}$  is clearly not closed as a subspace of  $X$ , yet it will be for the weak topology.

Given such a CW-structure on  $(X, A)$ , then denote by  $X_n$  the union of  $A$  and the cells of dimension  $\leq n$ ; we called this the  $n$ -skeleton of the CW-structure. The skeleta make up a chain of subspaces

$$X_\bullet = (A = X_{-1} \subset X_0 \subset X_1 \subset \dots),$$

with  $X = \cup_n X_n$ .

LEMMA 4.8. *For all  $n \geq -1$ ,  $X_n$  is closed in  $X$  and the CW-structure restricts to one on  $X_n$ . Moreover, any  $n$ -cell is open in  $X_n$  and is a connected component of  $X_n - X_{n-1}$  (so the entire CW structure is completely determined by the chain  $X_\bullet$ ).*

*Furthermore, any compact subspace of  $X$  hits only finitely many cells.*

PROOF. We check that if  $G \subset X_n$  has the property that its intersection with  $A$  and the closure of every cell of dimension  $\leq n$  is closed, then  $G$  is closed in  $X$ . This means that we must show that  $G$  meets the closure of any cell  $e$  of  $X_\bullet$  (not just of  $\dim \leq n$ ) in a closed subset of  $X$ . But  $\bar{e}$  is contained in a finite union of cells. Let  $e_1, \dots, e_r$  be the cells in  $X_n$  which meet  $\bar{e}$ . Then  $G \cap \bar{e} = (G \cap A) \cup (G \cap \bar{e}_1) \cup \dots \cup (G \cap \bar{e}_r)$  and since each of these terms is closed, so is  $G \cap \bar{e}$ . For  $G = X_n$  we find that  $X_n$  is closed in  $X$ .

We next prove that every  $n$ -cell  $e$  is open in  $X_n$ , or equivalently, that  $X_n - e$  is closed in  $X_n$ . If  $e'$  is a cell of  $X_n$ , then  $\dim(e') \leq n$  and so  $\bar{e}' \cap (X_n - e)$  is equal to  $\bar{e}'$  (when  $e' \neq e$ ) or  $\dot{e}'$  (when  $e' = e$ ) and so in either case closed in  $\bar{e}'$ . We also have that  $A \subset X_n - e$  and so by (W),  $X_n - e$  is closed in  $X_n$ .

Since the complement of  $e$  in  $X_n - X_{n-1}$  is a union of  $n$ -cells, and hence open in  $X_n - X_{n-1}$ ,  $e$  is also closed in  $X_n - X_{n-1}$ . So  $e$  is open and closed in  $X_n - X_{n-1}$ , and since  $e$  is connected, it must be a connected component of  $X_n - X_{n-1}$ .

Let now  $K \subset X$  be compact and denote by  $E$  the collection of cells of  $X$  which meet  $K$ . In order to show that  $E$  is finite, we choose for every  $e \in E$  an element  $p_e \in e \cap K$ , put  $P = \{p_e\}_{e \in E}$  and prove that  $P$  is finite. For every cell  $e$  of  $X$ ,  $\bar{e}$  meets only finitely many cells and so  $\bar{e} \cap P$  will be finite and hence closed in  $\bar{e}$ . Clearly  $P \cap A = \emptyset$  is also closed. By (W),  $P$  is then closed in  $X$ . For the same reason we have that for every  $p_e \in P$ ,  $P - \{p_e\}$  is closed in  $X$  so that  $\{p_e\}$  is open in  $P$ . Thus  $\{p_e\}_{e \in E}$  is an open covering of  $P$ . Since  $P$  is closed subset of the compact  $K$ ,  $P$  is compact as well and so  $P$  must be finite.  $\square$

Many space pairs admit a CW-structure. Here are a few examples.

EXAMPLE 4.9. The  $n$ -sphere  $S^n$  admits a CW-structure with only two cells: singleton (a 0-cell) and its complement (an  $n$ -cell). Another such structure is given by the chain

$$\emptyset = S^{-1} \subset S^0 \subset S^1 \subset \dots \subset S^n,$$

where  $S^k \subset S^n$  is defined by putting the last  $n - k$  coordinates equal to zero. Notice that  $S^k$  is obtained from  $S^{k-1}$  by attaching two  $k$ -cells whose closures are the upper hemisphere  $B_+^k$  and a lower hemisphere  $B_-^k$ .

EXAMPLE 4.10. We consider real projective  $n$ -space  $P^n$  as a quotient of  $S^n$  by identification of antipodes. Under this identification map  $S^k \subset S^n$  is mapped to  $P^k \subset P^n$ , and the composite

$$(B^n, S^{n-1}) \cong (B_+^n, S^{n-1}) \subset (S^n, S^{n-1}) \rightarrow (P^n, P^{n-1})$$

is a characteristic map. So the CW-structure for  $S^n$  determines one for  $P^n$ :  $P^k$  is obtained from  $P^{k-1}$  by attaching a  $k$ -cell and

$$\emptyset = P^{-1} \subset P^0 \subset P^1 \subset \dots \subset P^n$$

defines a CW-structure for  $P^n$ .

EXAMPLE 4.11. A point of complex projective  $n$ -space  $P_n(\mathbb{C})$  is given by the ratios of an  $(n+1)$ -tuple of complex numbers that are not all zero:  $[z^0 : z^1 : \dots : z^n]$ . We may regard  $P_{n-1}(\mathbb{C})$  as the subspace of  $P_n(\mathbb{C})$  defined by  $z^n = 0$ . Then  $P_n(\mathbb{C})$  is obtained from  $P_{n-1}(\mathbb{C})$  by attaching to it a  $2n$ -cel: if  $B^{2n}$  is the closed unit ball in  $\mathbb{C}^n$ , then

$$z = (z^0, \dots, z^{n-1}) \in B^{2n} \mapsto [z^0 : \dots : z^{n-1} : 1 - \|z\|^2]$$

is a characteristic map from  $(B^{2n}, S^{2n-1})$  onto  $(P_n(\mathbb{C}), P_{n-1}(\mathbb{C}))$ .

EXAMPLE 4.12. The torus  $T = S^1 \times S^1$  admits a CW-structure defined by

$$\{(1, 1)\} \subset \{1\} \times S^1 \cup S^1 \times \{1\} \subset T,$$

which has one 0-cell, two 1-cells and a 2-cell. A characteristic map for the 2-cell is the composite of a homeomorphism of  $B^2$  onto  $[0, 2\pi] \times [0, 2\pi]$  and the map  $(\alpha, \beta) \in [0, 2\pi] \times [0, 2\pi] \mapsto (e^{i\alpha}, e^{i\beta})$ .

EXAMPLE 4.13. The boundary  $X$  of a smooth solid body in  $\mathbb{R}^3$  with  $g$  holes (more precisely, a space homeomorphic to a compact connected orientable surface of genus  $g$ ) admits a CW-structure with one 0-cell,  $2g$  1-cells  $a_1, \dots, a_g, b_1, \dots, b_g$  and a single 2-cel. This follows from a description of  $X$  as a quotient space of the 2-ball  $B^2$ : divide its boundary  $S^1$  into  $4g$  equal arcs, labelled in counterclock wise order  $a_1, b_1, a'_1, b'_1, a_2, \dots, b'_g, a'_g$  and identify  $a_i$  with  $a'_i$  and  $b_i$  with  $b'_i$  in an orientation reversing manner.

Suppose  $(X, A)$  endowed with an oriented CW-structure  $X_\bullet$ . We associate to this structure a chain complex that in general is much smaller than the singular chain complex of  $(X, A)$ , yet has the same homology. This often makes the homology of  $(X, A)$  computable.

LEMMA 4.14. *We have  $H_l(X_k, X_{k-1}) = 0$  when  $l \neq k$ , whereas  $H_k(X_k, X_{k-1})$  maybe identified with the free abelian group generated by the (oriented)  $k$ -cells of  $X_\bullet$ . To be precise, the embedding of (resp. projection onto) the summand defined by the  $k$ -cell  $e$  is given by*

$$H_k(\bar{e}, \dot{e}) \hookrightarrow H_k(X_k, X_{k-1}) \twoheadrightarrow H_k(X_k, X_k - e) \cong H_k(\bar{e}, \dot{e})$$

(the composite is the identity).

PROOF. In case  $X$  has a single  $k$ -cell, then this is the content of Lemma 4.4, but the argument used there generalizes to the case when  $X$  has finitely many  $k$ -cells. The general case follows then from the fact that the support of a chain is compact and so lies in the union of finitely many cells (we omit the details, which are in fact not hard to supply).  $\square$

COROLLARY 4.15. *For  $p > k > q$ , the inclusions define isomorphisms*

$$H_k(X_p, X_{q-1}) \xleftarrow{\cong} H_k(X_p, A) \xrightarrow{\cong} H_k(X, A).$$

PROOF. The first isomorphism follows with induction on  $q$  (starting with  $q = 0$ ): the induction step is set (for  $q \geq 1$ ) by examining the long exact sequence of the triple  $(X_p, X_{q-1}, X_{q-2})$ :

$$H_k(X_{q-1}, X_{q-2}) \rightarrow H_k(X_p, X_{q-2}) \rightarrow H_k(X_p, X_{q-1}) \rightarrow H_{k-1}(X_{q-1}, X_{q-2})$$

and observing that since  $k, k-1 \neq q-1$ , the extremal terms are zero by Lemma 4.14. A similar argument shows with induction that for all  $n \geq p$  we have isomorphisms

$$H_k(X_p, A) \cong H_k(X_{p+1}, A) \cong H_k(X_{p+2}, A) \cong \cdots \cong H_k(X_n, A).$$

To see that this implies that  $H_k(X_p, A) \cong H_k(X, A)$ , we need to go back to the definition of our homology groups (and thus a bit beyond the fundamental properties derived in Section 2). The argument rests on the observation that if  $z \in C_k(X)$  is a  $k$ -chain, then the support of  $z$  is a finite number of simplices and hence contained in a compact set. So by Lemma 4.8 this support is contained in some  $X_n$  for some  $n \geq p$ . So if  $z$  defines a  $k$ -cycle for  $(X, A)$ , then it defines one for  $(X_n, A)$  for some  $n \geq p$ . In other words,  $H_k(X_p, A) \rightarrow H_k(X, A)$  is onto. If  $z$  is a  $k$ -chain on  $X_p$  and defines a  $k$ -coboundary for  $(X, A)$  so that  $z - \partial w$  has support in  $A$  for some  $(k+1)$ -chain  $w$  on  $X$ , then  $w$  has its support in some  $X_n$  with  $n \geq p$  and hence defines the zero element in  $H_k(X_n, A)$ . So  $H_k(X_p, A) \rightarrow H_k(X, A)$  is into as well.  $\square$

The triple  $(X_k, X_{k-1}, X_{k-2})$  gives rise to a boundary map

$$\partial_l : H_l(X_k, X_{k-1}) \rightarrow H_{l-1}(X_{k-1}, X_{k-2}),$$

which, in view of the preceding lemma is only of interest if  $l = k$ . We write  $C_k$  for  $H_k(X_k, X_{k-1})$ . This is the free abelian group generated by the  $e$ 's, where  $e$  runs over the  $k$ -cells.

PROPOSITION-DEFINITION 4.16. *The sequence*

$$C_\bullet : \cdots \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} C_0 \rightarrow 0$$

is a chain complex of free abelian groups, called the cellular chain complex of  $X_\bullet$ , whose  $k$ -th homology group may be identified with  $H_k(X, A)$ .

PROOF. The first clause follows from  $dd = 0$ . Consider the following section of the exact sequence of the triple  $(X_k, X_{k-1}, X_{k-2})$ :

$$H_k(X_{k-1}, X_{k-2}) \rightarrow H_k(X_k, X_{k-2}) \rightarrow C_k \xrightarrow{\partial_k} C_{k-1}.$$

By 4.14 the first term is zero, and so  $H_k(X_k, X_{k-2}) \cong \text{Ker}(\partial_k : C_k \rightarrow C_{k-1})$ . In the exact sequence of the triple  $(X_{k+1}, X_k, X_{k-2})$  we find

$$\begin{array}{ccccc} H_{k+1}(X_{k+1}, X_k) & \xrightarrow{\partial_{k+1}} & H_k(X_k, X_{k-2}) & \longrightarrow & H_k(X_{k+1}, X_{k-2}) \rightarrow 0, \\ \parallel & & \cong \downarrow & & \\ C_{k+1} & \xrightarrow{\partial_{k+1}} & \text{Ker}(\partial_k : C_k \rightarrow C_{k-1}) & & \end{array}$$

with the top row exact because  $H_k(X_{k+1}, X_k) = 0$  (by 4.14). It follows that

$$H_k(X_{k+1}, X_{k-2}) \cong H_k(X_k, X_{k-2}) / \partial_{k+1}(C_{k+1}) \cong H_k(C_\bullet).$$

By Corollary 4.15,  $H_k(X_{k+1}, X_{k-2})$  can be identified with  $H_k(X, A)$ .  $\square$

COROLLARY 4.17. *The homology of a complex projective space is given by*

$$H_k(P_n(\mathbb{C})) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ even and moreover } 0 \leq k \leq 2n; \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. By example 4.11,  $C_k$  is isomorphic to  $\mathbb{Z}$  if  $k$  is even,  $0 \leq k \leq 2n$ , and zero otherwise. It is clear that  $H_k(C_\bullet) = C_k$ .  $\square$

The usefulness of Proposition 4.16 would of course be greatly enhanced if it came with a recipe for computing the boundary operator of the cellular chain complex. Let us first make this problem explicit and then see what we can do. Given oriented cells  $e$  and  $e'$  with  $\dim e = \dim e' + 1$ , then we call the coefficient of  $e'$  in  $\partial e$  the *incidence number* of  $e$  and  $e'$ , and denote it by  $d(e, e')$ . So this is the matrix coefficient of the homomorphism  $\partial_k$ . Our goal is now to interpret the incidence number  $d(e, e')$  as the degree of a map. For this we assume given characteristic maps  $j : (B^{k+1}, S^k) \rightarrow (\bar{e}, \bar{e})$  and  $j' : (B^k, S^{k-1}) \rightarrow (\bar{e}', \bar{e}')$  that induce the given orientations in the sense that these are the images of the standard generator of  $H_{k+1}(B^{k+1}, S^k)$  resp.  $H_k(B^k, S^{k-1})$ .

PROPOSITION 4.18 (Supplement of Proposition 4.16). *The incidence number  $d(e, e')$  is the degree of the composite map*

$$S^k \xrightarrow{j|S^k} X_k \xrightarrow{r} X_k/(X_k - e') \xrightarrow{\cong} B^k/S^{k-1}.$$

To be precise, for  $k > 0$  it is the multiplier of the map  $H_k(S^k) \rightarrow H_k(B^k/S^{k-1}) \cong H_k(S^k)$ , whereas for  $k = 0$ , it is the one of  $\mathbb{Z} = \tilde{H}_0(S^0) \rightarrow H_0(B^0/S^{-1}) = \mathbb{Z}$ .

PROOF. For  $k > 0$  we consider the commutative diagram below.

$$\begin{array}{ccc} \mathbb{Z} \cong H_{k+1}(B^{k+1}, S^k) & \xrightarrow{j_*} & H_{k+1}(X_{k+1}, X_k) \\ \partial_{k+1} \downarrow \cong & & \partial_{k+1} \downarrow \\ H_k(S^k) & \longrightarrow & H_k(X_k, X_{k-1}) \xrightarrow{r_*} H_k(\bar{e}', \bar{e}') \cong H_k(B^k/S^{k-1}). \end{array}$$

According to Lemma 4.14,  $j_*$  maps 1 to  $e$  and  $r_*$  defines the projection onto  $\mathbb{Z}e'$ . This shows that the composite of the bottom maps is multiplication by  $d(e, e')$ . The proof for  $k = 0$  requires a minor adaption: notice that the vertical map on the left is then an isomorphism onto  $\tilde{H}_0(S^0) = \mathbb{Z}$  and that  $H_0(B^0, S^{-1}) = H_0(B^0) = \mathbb{Z}$ .  $\square$

EXAMPLE 4.19. Let  $P = X_0 \subset X_1 \subset X$  be the CW-structure in Example 4.13 on a compact connected oriented surface of genus  $g$ . Then  $\partial_1$  is the zero map (why?) and so is  $\partial_2 = 0$ : the attaching map for the 2-cell  $e$ ,  $f : S^1 \rightarrow X_1$ , traverses the 1-cell  $e'$  in  $X_1$  in both directions once, and so  $d(e, e') = 0$ . We conclude that  $H_k(X) \cong C_k$ , and so is  $\cong \mathbb{Z}$  in degrees 0 and 2,  $\cong \mathbb{Z}^{2g}$  in degree 1, and 0 otherwise. If we consider the characteristic maps for the 1-cells as singular 1-simplices (here we identify  $B^1$  with  $\Delta^1$ ), then these are 1-cycles, whose classes make up an integral basis of  $H_1(X)$ .

We can now also compute the homology of the real projective spaces:

PROPOSITION 4.20. *The homology of the real projective spaces is given by*

$$H_k(P^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } (k = n \text{ and } n \text{ odd}); \\ \mathbb{Z}/2 & \text{if } k \text{ odd and } 1 \leq k \leq n-1; \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let  $\mathbf{e}_k$  ( $k = 0, \dots, n$ ) be the oriented  $k$ -cell of  $P^n$  defined in Example 4.10. We show that  $\pm d(\mathbf{e}_{k+1}, \mathbf{e}_k) = 1 + (-1)^{k+1}$ ; the assertion then follows from Proposition 4.16. According to 4.18  $d(\mathbf{e}_{k+1}, \mathbf{e}_k)$  equals the degree of the composite

$$f_k : S^k \rightarrow P^k \rightarrow P^k/P^{k-1} \cong B_+^k/S^{k-1} \cong S^k,$$

in which the first map is antipodal identification, the third is induced by restricting the latter to the hemisphere  $B_+^k \subset S^k$  defined  $x_{k+1} \geq 0$  and the last map is the obvious one. We compute this degree by applying Proposition 3.15 to the last unit vector  $q$  in  $S^k$ . It is clear that  $f_k^{-1}(q) = \{q, -q\}$ . Now  $f_k$  maps the open hemisphere defined by  $x_{k+1} > 0$  homeomorphically onto  $S^k - \{-q\}$  and so  $\deg_q(f_k) = \pm 1$  (in fact it is equal to 1, but this will not matter in what follows) The restriction of  $f_k|_{B_-^k}$  is the composite of the antipodal map and  $f_k|_{B_+^k}$ . Since the antipodal map has degree  $(-1)^{k+1}$ , it follows that  $\deg_{-q}(f_k) = (-1)^{k+1} \deg_q(f_k)$ . So  $\deg(f_k) = (1 + (-1)^{k+1}) \deg_q(f_k)$ .  $\square$

PROBLEM 24. The  $n$ -dimensional projective space over the quaternions  $P_n(\mathbb{H})$ , is by definition the quotient of  $\mathbb{H}^{n+1} - \{0\}$  by the equivalence relation

$$(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n), \quad \lambda \in \mathbb{H} - \{0\}.$$

Prove that  $\emptyset \subset P_0(\mathbb{H}) \subset P_1(\mathbb{H}) \subset \dots \subset P_n(\mathbb{H})$  defines a CW-structure with exactly one cell in dimensions  $0, 4, 8, \dots, 4n$  and no others. Compute the homology of  $P_n(\mathbb{H})$ .

PROBLEM 25. Let  $(X, A)$  be a space pair with  $A$  arcwise connected. Suppose  $(X, A)$  equipped with a CW-structure  $X_\bullet$  such that for every cell  $e$  of  $\dim \geq 1$  we have  $\dot{e} \subset X_{\dim(e)-2}$ . Prove that  $H_k(X, A) = C_k(X_\bullet)$ .

PROBLEM 26. Let  $X_\bullet$  and  $Y_\bullet$  be CW complexes with finitely many cells and put  $(X \times Y)_k := \cup_{i=0}^k X_i \times Y_{k-i}$ .

(a) Prove that  $(X \times Y)_\bullet$  is a CW-complex.

(b) Suppose that  $X_\bullet$  and  $Y_\bullet$  are oriented. Describe how this (naturally) orients the cells  $(X \times Y)_\bullet$ .

(c) Compute the homology of  $S^k \times S^l$ .

(d) Try to describe the cellular chain complex of  $(X \times Y)_\bullet$  in terms of that of  $X_\bullet$  and  $Y_\bullet$ .

PROBLEM 27. The *Klein bottle* is obtained from the cylinder  $S^1 \times [-1, 1]$  by identifying  $(z, 1)$  with  $(\bar{z}, -1)$ . Give this space a CW-structure and compute its homology.

PROBLEM 28. Let  $C$  be the circle in the  $(x, y)$ -plane of  $\mathbb{R}^3$  centered at the origin and with radius  $\frac{1}{2}$ . Let  $Y \subset \mathbb{R}^3 - C$  be the union of the unit sphere  $S^2$  and the segment connecting the two poles  $(0, 0, 1)$  and  $(0, 0, -1)$ .

(a) Prove that  $Y \subset \mathbb{R}^3 - C$  is a homotopy equivalence.

(b) Give  $Y$  with a CW-structure and compute its homology.

PROBLEM 29. Let  $C \subset \mathbb{R}^3$  be the unit circle in the  $(x, y)$ -plane centered at the origin and  $C' \subset \mathbb{R}^3$  the unit circle in the  $(x, z)$ -plane centered at  $(1, 0, 0)$ . Let  $T$  resp.  $T'$  be the torus consisting of points having distance  $\frac{1}{2}$  to  $C$  resp.  $C'$ .

(a) Prove that  $T \cap T'$  is the union of  $\frac{1}{2}C$  and  $\frac{1}{2}C'$

(b) Make plausible that  $T \cup T' \subset \mathbb{R}^3 - (C \cup C')$  is a homotopy equivalence.

(c) Give  $T \cup T'$  a CW-structure with one 0-cell, two 1-cells and two 2-cells.



(d) Compute the homology of  $\mathbb{R}^3 - (C \cup C')$ .

PROBLEM 30. Let  $(X, A)$  be a space pair (Hausdorff as always in this chapter) that can be given a CW-structure. Prove that for  $A$  nonempty, the obvious map  $X \rightarrow X/A$  induces an isomorphism of  $H_k(X, A)$  onto  $\tilde{H}_k(X/A)$  for all  $k$ .

## 5. Cellular maps

We begin with a definition.

DEFINITION 5.1. Let  $(X, A)$  and  $(Y, B)$  be Hausdorff pairs endowed with CW-structures  $X_\bullet$  resp.  $Y_\bullet$ . We say that a map  $f : (X, A) \rightarrow (Y, B)$  is *cellular* with respect to these structures if  $f(X_k) \subset Y_k$  for all  $k$ .

The interest if this notion is in part explained by the following proposition. (We here assume that both cell complexes have been oriented.)

PROPOSITION 5.2. Let  $(X, A)$  and  $(Y, B)$  be Hausdorff pairs endowed with oriented CW-structures  $X_\bullet$  resp.  $Y_\bullet$ . A cellular map  $f : (X, A) \rightarrow (Y, B)$  determines a chain map  $C_\bullet(X_\bullet) \rightarrow C_\bullet(Y_\bullet)$  of cellular chain complexes, which on homology induces the map  $H_k(f) : H_k(X, A) \rightarrow H_k(Y, B)$ . For an oriented  $k$ -cell  $e$  of  $X_\bullet$ , an oriented  $k$ -cell  $e'$  of  $Y_\bullet$ , the coefficient of  $C_\bullet(f)(e)$  on  $e'$  is the degree of the map

$$\bar{e}/\dot{e} \subset X_k/X_{k-1} \rightarrow Y_k/Y_{k-1} \rightarrow Y_k/(Y_k - e') \cong \bar{e}'/\dot{e}'.$$

PROOF. Left to you. □

We next show that for  $(X, A)$  and  $(Y, B)$  as in Proposition 5.2 any map  $(X, A) \rightarrow (Y, B)$  is in a rather special way homotopic to a cellular one.

If a subspace  $X' \subset X$  is the union of the closures of a collection of cells and a subset of  $A$ , then it is clear that the cells in  $X'$  give  $(X', X' \cap A)$  a CW-structure  $X'_\bullet$ . We therefore call  $X'_\bullet$  a *subcomplex* of  $X_\bullet$ .

Any intersection of subcomplexes is subcomplex and so for every subspace  $X' \subset X$  there is a smallest subcomplex of  $X_\bullet$  that contains  $X'$ ; we call this the *cellular hull* of  $X'$  and we shall denote it by  $\text{Hull}(X')$ . In concrete terms: the closure of a cell  $\bar{e}$  belongs to  $\text{Hull}(X')$  precisely when there is a chain of cells  $e_0, e_1, \dots, e_k = e$  such that  $e_0 \cap X' \neq \emptyset$  and  $e_i \cap \bar{e}_{i-1} \neq \emptyset$  for  $i = 1, \dots, k$  (so  $\dim e_0 > \dim e_1 > \dots > \dim e_k = \dim e$ ). By property (C) the closure of a cell meets only a finite number of cells and so if  $X'$  meets finitely many cells, then  $\text{Hull}(X')$  has a finite number of cells. According to Lemma 4.8 this the case when  $X'$  is compact.

DEFINITION 5.3. Let  $f, f' : X \rightarrow Y$  be maps which coincide on  $A$ . A *homotopy from  $X$  to a space  $Y$  relative to  $A$*  is simply a homotopy  $F : [0, 1] \times X \rightarrow Y$  from  $f$  to  $f'$  with the property that  $F_t|_A$  is independent of  $t$ :  $F(t, a) = f(a)$  for all  $(t, a) \in [0, 1] \times A$ . If such a homotopy exists, then we say that  $f$  and  $f'$  are *homotopic relative to  $A$* .

PROPOSITION-DEFINITION 5.4. Suppose  $(X, A)$  and  $(Y, B)$  are Hausdorff pairs endowed with a CW-structure. Then for every map of pairs  $f : (X, A) \rightarrow (Y, B)$  there exists a homotopy  $F$  relative to  $A$  from  $f$  to a cellular map  $f'$  such that for every cell  $e$  of  $X_\bullet$  and every  $t \in [0, 1]$  we have  $F_t(e) \subset \text{Hull}(f(\bar{e}))$ . Such a map  $f'$  is called a *cellular approximation of  $f$* .

The proof is based on the following two lemmas. The first lemma implies that given a map  $f : X \rightarrow Y$ , then any homotopy of maps  $A \rightarrow Y$  which begins with  $f|_A$  can be extended to a homotopy of maps  $X \rightarrow Y$  which begins with  $f$ : if the former is given by  $F_A : A \times [0, 1] \rightarrow Y$ , then we can take for the latter  $\tilde{F}_A r : X \times [0, 1] \rightarrow Y$ , where  $\tilde{F}_A : X \times \{0\} \cup A \times [0, 1] \rightarrow Y$  is the union of  $F_A$  and  $f : X \times \{0\} \cong X \rightarrow Y$  (and  $r$  is as in the lemma).

LEMMA 5.5. *There is a retraction  $r : X \times [0, 1] \rightarrow A \times [0, 1] \cup X \times \{0\}$  such that for every cell  $e$  of  $X$ ,  $r(e \times [0, 1]) \subset \text{Hull}(e)$*

PROOF. We first construct for every cell  $e$  a retract

$$r_e : \bar{e} \times [0, 1] \rightarrow \dot{e} \times [0, 1] \cup \bar{e} \times \{0\}.$$

Choose a characteristic map  $j : (B^n, S^{n-1}) \rightarrow (\bar{e}, \dot{e})$ . We view  $B^n \times [0, 1]$  as a subspace of  $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$  and let  $\rho : B^n \times [0, 1] \rightarrow S^{n-1} \times [0, 1] \cup B^n \times \{0\}$  be the retract given by projection from  $(0, \dots, 0, 2)$ . Then the composite

$$\rho(j \times \mathbf{1}_{[0,1]}) : B^n \times [0, 1] \rightarrow \dot{e} \times [0, 1] \cup \bar{e} \times \{0\}$$

factors through a retract  $r_e$  as above.

Now let  $e$  run over all the  $k$ -cells of  $X_\bullet$ . Then the  $r_e$ 's combine to define a retract  $r_k : X_k \times [0, 1] \rightarrow X_{k-1} \times [0, 1] \cup X_k \times \{0\}$ . These in turn determine a retract  $r : X \times [0, 1] \rightarrow A \times [0, 1] \cup X \times \{0\}$  which on  $X_k$  is given by  $r_0 r_1 \cdots r_k$ . This  $r$  is as desired.  $\square$

LEMMA 5.6. *A map  $f : (B^k, S^{k-1}) \rightarrow (Y, Y_{k-1})$  is homotopic relative to  $S^{k-1}$  to a map whose image is contained in  $Y_k$  under a homotopy whose image lies in the cellular hull of  $f(B^k)$ .*

PROOF. Without loss of generality we can assume that  $Y$  equals the cellular hull of  $f(B^k)$ . Now  $Y$  has only finitely many cells so that  $Y = Y_l$  for some  $l \geq k$ . It then suffices to show that if  $l > k$ , then  $f$  is homotopic relative to  $S^{k-1}$  with a map whose image lands in  $Y_{l-1}$ . We do this cell by cell, to be precise, we show that for every  $l$ -cel  $e$  of  $Y_\bullet$  there is a homotopy relative to  $S^{k-1}$  from  $f$  to a map whose image lands in  $Y_l - e$ .

If the image  $f$  misses a point  $p \in e$ , then a homotopy is obtained as follows. Choose a characteristic map  $j : B^l \rightarrow \bar{e}$  for  $e$  with  $j(0) = p$  and let  $r : Y_l - \{p\} \rightarrow Y_l - e$  be the retract which assigns to  $x \in e - \{p\}$  the image of  $j^{-1}(x)/|j^{-1}(x)| \in S^{l-1}$  under  $j$  (and is the identity on  $Y_l - e$ ). Then

$$r_t(x) = \begin{cases} y & \text{als } y \in Y_l - e; \\ j\left(\frac{(1-t)j^{-1}(x) + tj^{-1}(x)}{|j^{-1}(x)|}\right) & \text{als } x \in e - \{p\}, \end{cases}$$

is a homotopy relative to  $Y_l - e$  of the identity map of  $Y - \{p\}$  to  $r$ . So  $r_t f$  yields a homotopy relative to  $S^{k-1}$  from  $f$  to  $r f$  and the image of the latter lands in  $Y_l - e$ .

Hence it suffices to show that  $f$  is homotopic relative to  $S^{k-1}$  to a map  $g$  whose image does not contain  $e$ . The idea for finding such a homotopy is to approximate  $f$  in a piecewise linear manner on preimage of a compact ball in  $e$ .

This leads the following construction. By means of characteristic map we identify  $e$  with the interior of  $B^l$  and write for  $0 \leq r < 1$ ,  $\bar{e}(r)$  for the (image of the) closed  $r$ -ball in  $B^l$ .

Let  $\phi : Y \rightarrow [0, 1]$  be the function which on  $e$  is  $x \mapsto \max\{2|x| - 1, 0\}$  and constant 1 on  $B$ . Notice that the zero set of  $\phi$  is  $\bar{e}(\frac{1}{2})$  and hence is disjoint with

$Y_{k-1}$ . Choose a simplex in  $\mathbb{R}^k$  that contains  $B^k$  and apply repeated barycentric subdivision to this simplex, so that every simplex  $\sigma$  of this subdivision for which  $f(\sigma)$  meets  $\bar{e}(\frac{3}{4})$ ,  $f(\sigma)$  has diameter  $< \frac{1}{4}$ . So if  $K$  denotes the union of these simplices, then  $K$  is compact,  $f(K) \subset e$  and  $f(K)$  contains  $\bar{e}(\frac{3}{4})$ . We also observe that the boundary  $\partial K$  of  $K$  in  $B^k$  is the union of simplices  $\sigma$  with the property that  $f(\sigma)$  does not meet  $\bar{e}(\frac{1}{2})$  (or equivalently, on which  $\phi$  is identically zero). A map  $f_K : K \rightarrow e$  is given by stipulating that it be affine-linear on the simplices and equal to  $f$  on the vertices. Define  $g : B^k \rightarrow Y$  by

$$g(y) = \begin{cases} f(y) & \text{if } y \notin K, \\ \phi f(y) \cdot f(y) + (1 - \phi f(y)) \cdot f_K(y) & \text{if } y \in K. \end{cases}$$

This map is continuous for if  $y \in \partial K$ , then  $\phi f(y) = 0$  and hence  $g(y) = f(y)$ . It is homotopic to  $f$  relative to  $S^{k-1}$  by means of the homotopy  $(1-t)f + tg$ ,  $0 \leq t \leq 1$ .

We finish the proof by noting that  $g^{-1}\bar{e}(\frac{1}{4}) = f_K^{-1}\bar{e}(\frac{1}{4})$  and hence is contained in a finite union of simplices of  $\dim \leq l < k$ . So there is indeed a  $p \in e$  (in fact  $p \in \bar{e}(\frac{1}{4})$ ) which is not in  $g(B^k)$ .  $\square$

**PROOF OF PROPOSITION 5.4.** We assume (by induction) that a cellular approximation  $F_{n-1} : X_{n-1} \times [0, 1] \rightarrow Y_{n-1}$  of  $f|X_{n-1}$  has been constructed. The induction step consists in extending this to a cellular approximation  $F_n$  of  $f|X_n$ . We let  $F'_n : X_n \times \{0\} \cup X_{n-1} \times [0, 1] \rightarrow Y$  be the union of  $F_{n-1}$  and the map  $f$  on  $X_n \times \{0\}$ . By invoking 5.5, we find that  $F'_n$  is extendable to a homotopy from  $f|X_n$  to a map  $f'_n$  which has all the desired properties (so in particular,  $f'_n(X_{n-1}) \subset Y_{n-1}$ ), except perhaps that the  $n$ -cells of  $X_n$  might all not be mapped to  $Y_n$ . This however is remedied by applying Lemma 5.6 to the  $n$ -cells of  $f'_n$ .  $\square$

For the following application we need a bit of linear algebra. Let us fix a field  $K$ . It can be arbitrary, but for what follows, the most relevant case is when  $K$  is one of the prime fields  $\mathbb{Q}$  or  $\mathbb{F}_p$  ( $p$  prime).

Let  $V$  be a vector space of finite dimension  $n$  over  $K$  and  $f : V \rightarrow V$  a linear map. If  $e_1, \dots, e_n$  is a basis of  $V$  and  $(f_i^j)_{i,j}$  the matrix of  $f$  with respect to that basis:  $f(e_i) = \sum_{j=1}^n f_i^j e_j$ , then the sum of the diagonal coefficients,  $\sum_{i=1}^n f_i^i$  equals minus the coefficient of  $t^{n-1}$  in the characteristic polynomial  $\det(t1_V - f)$  of  $f$ , and hence is independent of the basis. We call this the *trace* of  $f$  and denote it  $\text{Tr}(f) \in K$ .

If  $V' \subset V$  is a subspace preserved by  $f$ , then  $f$  restricts to a linear map  $f' : V' \rightarrow V'$  and hence induce a linear map  $\bar{f}$  in  $\bar{V} := V/V'$ . We then have  $\text{Tr}(f) = \text{Tr}(f') + \text{Tr}(\bar{f})$ : this is clear if you choose the basis  $e_1, \dots, e_n$  in such a manner that  $e_1, \dots, e_{n'}$  is a basis of  $V'$  (with  $n' := \dim V'$ ) so that  $e_{n'+1}, \dots, e_n$  projects onto a basis of  $\bar{V}$ , for then the matrices of  $f'$  resp.  $\bar{f}$  are  $(f_i^j)_{1 \leq i,j \leq n'}$  resp.  $(f_i^j)_{n'+1 \leq i,j \leq n}$ .

Let  $C_\bullet$  be a complex of finite dimensional vector spaces with only finitely many nonzero terms and  $\phi_\bullet : C_\bullet \rightarrow C_\bullet$  a chain map. We here make the natural assumption that all the maps, i.e., the  $d_l$ 's and the  $\phi_l$ 's, are  $K$ -linear. Then the homology groups are in fact  $K$ -vector spaces and  $H_l(\phi_\bullet) : H_l(C_\bullet) \rightarrow H_l(C_\bullet)$  is  $K$ -linear.

**LEMMA 5.7.** *We then have  $\sum_l (-1)^l \text{Tr}(\phi_l) = \sum_l (-1)^l \text{Tr}(H_l(\phi_\bullet))$ .*

PROOF. Consider the exact sequences

$$\begin{aligned} 0 \rightarrow Z_l(C_\bullet) \rightarrow C_l \xrightarrow{d} B_{l-1}(C_\bullet) \rightarrow 0 \\ 0 \rightarrow B_l(C_\bullet) \rightarrow Z_l(C_\bullet) \rightarrow H_l(C_\bullet) \rightarrow 0 \end{aligned}$$

and note that  $\phi_l$  induces in each of these a linear map. If we denote the trace of  $\phi_l$  in  $Z_l(C_\bullet)$  resp.  $B_l(C_\bullet)$  by  $z_l$  resp.  $b_l$ , then we see that

$$\begin{aligned} \text{Tr}(\phi_l) &= z_l + b_{l-1}, \\ z_l &= b_l + \text{Tr}(H_l(\phi_\bullet)), \end{aligned}$$

so that  $\text{Tr}(\phi_l) - \text{Tr}(H_l(\phi_\bullet)) = b_{l-1} + b_l$ . If we now take the alternating sum we get

$$\sum_l (-1)^l (\text{Tr}(\phi_l) - \text{Tr}(H_l(\phi_\bullet))) = \sum_l (-1)^l (b_l + b_{l-1}) = 0. \quad \square$$

If  $A$  is a finitely generated abelian group, then  $K \otimes A$  is a finite dimensional  $K$ -vector space. In fact, if you write  $A$  in terms of generators and relations, then  $A_K := K \otimes A$  is described by the same generators and relations, but now as  $K$  vector spaces. The rules governing this operation are:  $(A \oplus B)_K = A_K \oplus B_K$ ,  $(\mathbb{Z})_K = K$ , and if  $N$  is a positive integer  $N$ , then  $(\mathbb{Z}/N)_K = 0$ , unless  $K \supset \mathbb{F}_p$ , where  $p$  is a prime number which divides  $N$ : in that case  $(\mathbb{Z}/N)_K$  is identified with  $K$ . For instance, the dimension of  $A_{\mathbb{Q}}$  is the rank of  $A$ . Or if  $A$  is free on  $n$  generators  $e_1, \dots, e_n$ , then  $A_K$  has basis  $(1 \otimes e_1, \dots, 1 \otimes e_n)$ . (More details on this in the next chapter.)

A homomorphism  $f : A \rightarrow B$  of abelian groups induces a linear map  $f_K : A_K \rightarrow B_K$ . In case  $f : A \rightarrow A$  is an endomorphism, then we define  $\text{Tr}_K(f)$  as the trace of the  $f_K$ . It follows from the definition that this trace lies in the image of  $\mathbb{Z}$  in  $K$ .

Let  $(X, A)$  be a space pair for which the total homology  $\bigoplus_l H_l(X, A)$  is finitely generated. This is for instance the case when  $(X, A)$  can be endowed with a finite CW-structure. We then define the  $K$ -Lefschetz number of  $f$  by:

$$L_K(f) := \sum_l (-1)^l \text{Tr}(H_l(f)_K : H_l(X, A)_K \rightarrow H_l(X, A)_K) \in K.$$

If we take  $f$  to be the identity, then we get  $\sum_l (-1)^l \dim_K H_l(X, A)_K$ . This is called the  $K$ -Euler characteristic of  $(X, A)$  and is denoted  $\chi_K(X, A)$ .

In case  $K = \mathbb{Q}$ , we often write  $L(f)$  resp.  $\chi(X)$  and call it just the Lefschetz number resp. Euler characteristic; these are then integers.

**COROLLARY 5.8.** *Let  $(X, A)$  be endowed with a finite CW-structure  $X_\bullet$  and denote by  $C_\bullet$  the corresponding cellular chain complex. If  $f : (X, A) \rightarrow (X, A)$  is a cellular map with respect to that structure, then  $L_K(f) = \sum_l (-1)^l \text{Tr}(C_l(f)_K)$ . Moreover;  $L_K(f) = 0$  if  $f(e) \cap e = \emptyset$  for every cell  $e$ .*

PROOF. The first assertion is indeed a consequence of 4.16 and the Propositions 5.2 en 5.7. If  $f(e) \cap e = \emptyset$ , then according to 5.2, the diagonal entry of the matrix of  $C_l(f_\bullet)$  associated to  $e$  is zero.  $\square$

Our goal is to sharpen the preceding corollary to:

**THEOREM 5.9.** *Let  $X$  be a compact Hausdorff space  $X$  which admits the structure of a finite CW complex. Then for any map  $f : X \rightarrow X$  without fixed point,  $L_K(f) = 0$ .*

The Lefschetz number of any map  $f : S^n \rightarrow S^n$  is  $1 + (-1)^n \deg(f)$ . So if  $f : S^n \rightarrow S^n$  is without fixed points, then has degree  $(-1)^{n+1}$ . We thus recover the fact that the antipodal map has degree  $(-1)^{n+1}$ .

For the proof of Theorem 5.9 we need some auxiliary results. We now show that a given CW-structure can be refined as much as we want.

LEMMA 5.10. *Let  $X$  be Hausdorff space which admits the structure of a finite CW-complex. Then for every open covering  $\mathcal{U}$  of  $X$ , there exists a CW-structure on  $X$  subordinate to  $\mathcal{U}$  in the sense that the cellular hull of every cell is contained in a member of  $\mathcal{U}$ .*

PROOF. Let  $X$  be endowed with a finite CW-structure  $X_\bullet$  and suppose with induction on  $k$ , that the cellular hull of every cell  $e'$  of dimension  $< k$  is contained in a member  $U_{e'}$  of  $\mathcal{U}$ .

The induction step is taken by refining the cellular decomposition on the  $k$ -cells only. We identify  $\Delta^k$  with  $B^k$  so that a characteristic map for a  $k$ -cell  $e$  is given by  $j : \Delta^k \rightarrow \bar{e}$ . The barycentric subdivision of  $\Delta^k$  defines a CW-structure on  $\Delta^k$  (the  $l$ -skeleton is the union of the  $l$ -simplices of this subdivision). With the help of Corollary 2.14 we can iterate this barycentric subdivision until each of its members  $\sigma$  have the following property:

- (i)  $j(\sigma)$  is contained in a member of  $\mathcal{U}$ ,
- (ii) if  $j(\sigma)$  meets a cell  $e'$  of  $X_{k-1}$ , then  $j(\sigma) \subset U_{e'}$ .

Now refine the given cellular structure on  $X$  on  $e$  by taking there as cells the images under  $j$  of the relatively open simplices (of dimension  $\leq k$ ) of  $\Delta^k - \dot{\Delta}^k$  of the iterated barycentric subdivision obtained above. We do this for every  $k$ -cell  $e$  and thus find a new cellular structure on  $X$  with the desired property.  $\square$

PROOF OF 5.9. We first prove that for every  $x \in X$  there exists an open  $U_x \ni x$  such that  $f^{-1}U_x \cap U_x = \emptyset$ . We have  $x \notin f^{-1}(x)$  by assumption. Since  $X$  is compact Hausdorff, there exist disjoint neighborhoods  $U \ni x$  and  $V \supset f^{-1}(x)$ . Then  $X - V$  is closed and hence compact. Since  $X$  is compact Hausdorff,  $f(X - V)$  must be closed as well and so  $U' := X - f(X - V)$  is open. It is clear that  $f^{-1}U' \subset V$ . Since  $f^{-1}(x) \subset V$ , we have  $x \in U'$ . So  $U_x := U \cap U'$  is as desired. We first apply Lemma 5.10 to the covering  $\mathcal{U} = \{f^{-1}U_x\}_{x \in X}$  and thus find a CW-structure on  $X$  with the property that for every cell  $e$ ,  $\bar{e} \subset f^{-1}U_x$  for some  $x \in X$ . This implies that  $f(\bar{e})$  is disjoint with  $\bar{e}$ .

By further refining the CW-structure we can even arrange that for every cell  $e$ , the cellular hull of  $f(\bar{e})$  is disjoint with  $\bar{e}$ . A cellular approximation  $f'$  of  $f$  then has the property that  $f'(e) \cap e = \emptyset$ . By Proposition 5.8, we then have  $L(f') = 0$ . Since  $f$  and  $f'$  are homotopic, they have the same Lefschetz number and so  $L(f) = 0$ .  $\square$

PROBLEM 31. Let  $(X, A)$  and  $(Y, B)$  be Hausdorff pairs endowed with a CW-structure and let  $f : (X, A) \rightarrow (Y, B)$  be a cellular map. We denote by  $g : A \rightarrow B$  the restriction of  $f$  and we consider  $X$ ,  $Y$  and  $Z_g$  as subspaces of the mapping cylinder  $Z_f$  (see problem 12).

(a) Show that  $(Z_f, Z_g)$  has a CW-structure whose cells are: the cells of  $\{1\} \times X$ , the cells of  $Y$  and for every cell  $e$  of  $X$ , the image of  $(0, 1) \times e$  in  $Z_f$ . In particular,  $(X, A)$  and  $(Y, B)$  become subcomplexes of  $(Z_f, Z_g)$ .

(b) Show that the cellular chain complex of  $(Z_f, X \cup Z_g)$  can be expressed entirely in terms of  $C_\bullet(f) : C_\bullet(X_\bullet) \rightarrow C_\bullet(Y_\bullet)$ . More precisely, describe an isomorphism

between the group of cellular  $k$ -chains of  $(Z_f, X \cup Z_g)$  with  $C_{k-1}(X_\bullet) \oplus C_k(Y_\bullet)$  so that the boundary map corresponds to

$$(u, v) \mapsto (\partial u, \partial v + (-1)^{k-1} C_{k-1}(f)(u)).$$

PROBLEM 32. Let  $(X, A)$  be a space pair and  $f : (X, A) \rightarrow (X, A)$  a map. We suppose that  $\sum_l \text{rang } H_l(X)$  and  $\sum_l \text{rang } H_l(A)$  are finite. Prove that  $\sum_l \text{rang } H_l(X, A)$  is also finite and that  $L(f|_X) = L(f) + L(f|_A)$ , where  $f|_X : X \rightarrow X$  resp.  $f|_A : A \rightarrow A$  are restrictions of  $f$ .

PROBLEM 33. Let  $T = \mathbb{R}^2/\mathbb{Z}^2$  and  $k \in \mathbb{Z}$ . Since scalar multiplication by  $k$  sends the equivalence class  $x + \mathbb{Z}^2$  to the equivalence class  $kx + \mathbb{Z}^2$ , this induces a continuous map  $m_k : T \rightarrow T$ . Determine the action of  $m_k$  on the homology of  $T$ , and prove that  $L(m_k) = 1 - 2k + k^2$ .

## 6. First steps in Homological Algebra

In this section we work with abelian groups only. Maps between such groups are always assumed to be homomorphisms (= linear maps). These are the objects and morphisms of a category, which we denote by  $\mathcal{Ab}$ . (A reader familiar with some basic commutative algebra will observe that part of the discussion below makes sense for modules over a commutative ring and that virtually all the assertions retain their validity if that ring is a principal ideal domain.)

Recall that an abelian group  $M$  is said to be *free* if there exists a subset  $B \subset M$  such that every element of  $M$  can be written in exactly one way as a finite sum  $\sum_i r_i b_i$  with  $b_i \in B$  and  $r_i \in \mathbb{Z}$  ( $B$  is then called a *basis* of  $M$ ). In other words, if the natural map  $\mathbb{Z}^{(B)} \rightarrow M$  is an isomorphism. So if  $B$  has only finitely many, say  $k$ , elements, then  $M \cong \mathbb{Z}^k$ .

Free abelian groups have the following fundamental property.

LEMMA 6.1. *Let  $f : M \rightarrow N$  be a surjective homomorphism and  $F$  a free abelian group. Then for every homomorphism  $g : F \rightarrow N$  there exists a homomorphism  $\tilde{g} : F \rightarrow M$  such that  $f\tilde{g} = g$ . In other words,*

$$f_* : \text{Hom}(F, M) \rightarrow \text{Hom}(F, N), \quad h \mapsto fh$$

*is onto. (Notice that  $f_*$  is a homomorphism of groups.)*

PROOF. If  $B$  is a basis for  $F$ , then choose for every  $b \in B$  an element  $m_b \in M$  whose image under  $f$  equals  $g(b)$  (such an  $m_b$  exists since  $f$  is onto), and define  $\tilde{g}$  by  $\tilde{g}(\sum_i r_i b_i) = \sum_i r_i m_{b_i}$ .  $\square$

We will also need:

LEMMA 6.2. *Any subgroup of a free abelian group is free.*

PROOF. We only give a proof in case the free group is finitely generated and then prove the lemma with induction on its rank. (The general case is based on transfinite induction.) So let  $M \subset \mathbb{Z}^k$  be a subgroup. If  $k = 0$ , then there is nothing to show. Assume  $k > 0$  and the lemma proved for smaller values of  $k$ . Let  $p : \mathbb{Z}^k \rightarrow \mathbb{Z}$  be the last component. Then  $M \cap \text{Ker}(p) = M \cap \mathbb{Z}^{k-1}$  is free by induction, with basis  $(b_1, \dots, b_l)$ , say. If  $M \subset \mathbb{Z}^{k-1}$ , we are done. Otherwise  $p(M)$  is a nonzero subgroup of  $\mathbb{Z}$ , hence has a nonzero generator  $0 \neq r \in \mathbb{Z}$ . Choose  $b_0 \in M$  with  $p(b_0) = r$ . Then you easily check that  $(b_0, \dots, b_l)$  is a basis for  $M$ .  $\square$

Associated to a pair  $(M, N)$  of abelian groups is their *tensor product*  $M \otimes N$ , which we may define as the quotient of  $\mathbb{Z}^{(M \times N)}$  (the group of formal finite integral linear combinations of elements of  $M \times N$ ) by the subgroup generated by the expressions

$$\begin{aligned} (m_1 + m_2, n) - (m_1, n) - (m_2, n) \\ (m, n_1 + n_2) - (m, n_1) - (m, n_2). \end{aligned}$$

The image of  $(m, n)$  in  $M \otimes N$  is denoted  $m \otimes n$ . So an arbitrary element of  $M \otimes N$  can be written as a finite sum  $\sum_i m_i \otimes n_i$ . The definition forces the following identities

$$\begin{aligned} (m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n, \\ m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2. \end{aligned}$$

We also note that for any  $r \in \mathbb{Z}$ ,  $rm \otimes n = m \otimes rn$ . We list a few basic properties.

- (1)  $M \otimes \mathbb{Z} \cong M$ , to be precise,  $m \in M \mapsto m \otimes 1$  is an isomorphism.
- (2)  $\mathbb{Z}^k \otimes \mathbb{Z}^l$  is isomorphic to  $\mathbb{Z}^{kl}$ : if  $(e_1, \dots, e_n)$  denotes the natural basis of  $\mathbb{Z}^n$ , then  $(e_i \otimes e_j)_{1 \leq i \leq k, 1 \leq j \leq l}$  is a basis of  $\mathbb{Z}^k \otimes \mathbb{Z}^l$ .
- (3) (generalizes (2)) For sets  $A$  and  $B$ , we have  $\mathbb{Z}^{(A)} \otimes \mathbb{Z}^{(B)} \cong \mathbb{Z}^{(A \times B)}$ .
- (4) A pair of homomorphisms  $f : M \rightarrow N$ ,  $f' : M' \rightarrow N'$  induces a homomorphism  $f \otimes f' : M \otimes M' \rightarrow N \otimes N'$  (this makes  $\otimes : \mathcal{A}b \times \mathcal{A}b \rightarrow \mathcal{A}b$  a functor).
- (5) The tensor product is associative and distributive in the sense that given abelian groups  $M$ ,  $N$  and  $P$ , then we have we isomorphisms

$$\begin{aligned} (M \otimes N) \otimes P &\cong M \otimes (N \otimes P), (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p), \\ (M \oplus N) \otimes P &\cong (M \otimes P) \oplus (N \otimes P), (m, n) \otimes p \mapsto (m \otimes p, n \otimes p), \\ (M \otimes (N \oplus P)) &\cong (M \otimes N) \oplus (M \otimes P), m \otimes (n, p) \mapsto (m \otimes n, m \otimes p). \end{aligned}$$

Furthermore, we the isomorphism

$$M \otimes N \cong N \otimes M, \quad m \otimes n \mapsto n \otimes m.$$

The following property in a sense characterizes the tensor product.

- (6) The map  $u : M \times N \rightarrow M \otimes N$ ,  $(m, n) \mapsto m \otimes n$  is *bilinear* (i.e., it is linear in each variable separately), and is universal for that property in the sense that any bilinear map  $f : M \times N \rightarrow P$  to a third abelian group  $P$  factors in exactly one way over  $u$ : there is unique homomorphism  $\tilde{f} : M \otimes N \rightarrow P$  such that  $f = \tilde{f}u$ , in other words,  $\tilde{f}(m \otimes n) = f(m, n)$ .

(We remark that the set of bilinear maps  $M \times N \rightarrow P$  is in an evident manner an abelian group  $\text{Bil}(M, N; P)$ . So the last property tells us that the assignment  $P \mapsto \text{Bil}(M, N; P)$  defines a functor  $\mathcal{A}b \rightarrow \mathcal{A}b$  which is represented by  $M \otimes N$ .)

LEMMA 6.3. *Let  $N$  be an abelian group and*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

*a sequence of abelian groups. If the sequence is exact, then so is the sequence*

$$A \otimes N \xrightarrow{\alpha \otimes 1} B \otimes N \xrightarrow{\beta \otimes 1} C \otimes N \longrightarrow 0.$$





$\psi_\bullet$  is chain homotopic to  $\phi_\bullet$ : there exists a homomorphism  $h : P_0 \rightarrow P'_1$  such that  $(\phi_0 - \psi_0, \phi_1 - \psi_1) = (d'h, hd)$ .

PROOF. Since  $u : P_0 \rightarrow M$  is onto and  $P_0$  free, the map  $fu : P_0 \rightarrow M'$  lifts to a homomorphism  $\phi_0 : P_0 \rightarrow P'_0$ . The latter  $\phi_0$  maps  $\text{Ker}(u)$  to  $\text{Ker}(u')$  and so we let  $\phi_1$  be this restriction.

Let  $\psi_\bullet$  be another solution and put  $\chi_\bullet := \phi_\bullet - \psi_\bullet$ . Then  $u'\chi_0 = (f - f)u = 0$ , and so the image of  $\chi_0$  is contained in  $\text{Ker}(u') \cong P'_1$ . Hence there exists a homomorphism  $h : P_0 \rightarrow P'_1$  such that  $d'h' = \chi_0$ . We now have  $d'\chi_1 = \chi_0d = d'hd$  and since  $d'$  is injective it follows that  $\chi_1 = hd$ .  $\square$

We use this to define a group that measures the failure of the exactness of the tensor product. Let  $M$  and  $N$  be abelian groups. Choose a short free resolution  $u : P_\bullet \rightarrow M$  of  $M$  and consider the sequence

$$P_1 \otimes N \rightarrow P_0 \otimes N \rightarrow M \otimes N \rightarrow 0$$

According to 6.3 this is exact, but the homomorphism  $P_1 \otimes N \rightarrow P_0 \otimes N$  need not be injective.

PROPOSITION-DEFINITION 6.6. *The kernel of  $P_1 \otimes N \rightarrow P_0 \otimes N$  is independent of the choice of a short free resolution in the sense that for two choices the associated kernels are naturally identified; we call it the torsion product of the pair  $(M, N)$  and denote it  $M \star N$ .*

PROOF. If  $P'_\bullet \rightarrow M$  is another short free resolution, then we have by 6.5 a chain map  $\phi_\bullet$  from  $P_\bullet$  to  $P'_\bullet$ . So  $\phi_1 \otimes 1$  maps  $K := \text{Ker}(P_1 \otimes N \rightarrow P_0 \otimes N)$  to  $K' := \text{Ker}(P'_1 \otimes N \rightarrow P'_0 \otimes N)$ . The resulting restriction  $K \rightarrow K'$  is unique, for the last part of 6.5 also implies that another choice of chain map yields the same restriction to  $K$ . In the same way we find a unique homomorphism  $K' \rightarrow K$ . The composites  $K \rightarrow K' \rightarrow K$  and  $K' \rightarrow K \rightarrow K'$  have the same unicity property and so must be equal to the identity maps of  $K$  resp.  $K'$ .  $\square$

We record part of this discussion as a corollary:

COROLLARY 6.7. *Formation of the torsion product enjoys the following properties.*

**Exact sequence:** *Given a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , with  $A$  and  $B$  free, then for every abelian group  $N$  the sequence*

$$0 \rightarrow C \star N \rightarrow A \otimes N \rightarrow B \otimes N \rightarrow C \otimes N \rightarrow 0.$$

*is exact.*

**Acyclicity of free abelian groups:** *If  $M$  is free, then  $M \star N = 0$ .*

**Functoriality:** *Two homomorphisms  $f : M \rightarrow M'$ ,  $g : N \rightarrow N'$  of abelian groups determine a homomorphism  $f \star g : M \star N \rightarrow M' \star N'$ .*

PROOF. Only the functoriality needs a proof. As this follows easily from Theorem 6.5, it is left to you.  $\square$

We shall later see that  $M \star N$  is symmetric in  $M$  and  $N$ .

EXAMPLE 6.8. We take  $M = \mathbb{Z}/(m)$  and  $N = \mathbb{Z}/(n)$  with  $n, m > 0$ . If we apply the preceding to the short free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m \times} \mathbb{Z} \longrightarrow \mathbb{Z}/(m) \longrightarrow 0,$$

then we find that  $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n) = \text{Coker}(\mathbb{Z}/(n) \xrightarrow{m \times} \mathbb{Z}/(n))$  and  $\mathbb{Z}/(m) \star \mathbb{Z}/(n) = \text{Ker}(\mathbb{Z}/(n) \xrightarrow{m \times} \mathbb{Z}/(n))$ . It follows that  $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n) = \mathbb{Z}/(m, n)$  and

$$\mathbb{Z}/(m) \star \mathbb{Z}/(n) = \{x \in \mathbb{Z} \mid mx \in n\mathbb{Z}\} / n\mathbb{Z} = \frac{n}{(n, m)} \mathbb{Z} / n\mathbb{Z} \cong \mathbb{Z}/(n, m).$$

Let  $C_\bullet$  and  $C'_\bullet$  be chain complexes. We define another chain complex, their *tensor product*  $C_\bullet \otimes C'_\bullet$ , as follows:

$$(C_\bullet \otimes C'_\bullet)_n = \bigoplus_{k+l=n} C_k \otimes C'_l$$

and the boundary map is given by

$$d(c \otimes c') = dz \otimes c' + (-1)^k c \otimes dc', \quad c \in C_k, c' \in C.$$

It is easily checked that  $dd = 0$ . Two homomorphisms  $\phi_\bullet : C_\bullet \rightarrow D_\bullet$  and  $\phi'_\bullet : C'_\bullet \rightarrow D'_\bullet$  determine in an evident manner a homomorphism  $\phi_\bullet \otimes \phi'_\bullet : C_\bullet \otimes C'_\bullet \rightarrow D_\bullet \otimes D'_\bullet$ .

The interchange map yields an isomorphism of  $C_k \otimes C'_l$  onto  $C'_l \otimes C_k$  as groups, but this does in general not define a chain map. For this we need to modify it with signs: one checks that the map  $c \otimes c' \in C_k \otimes C'_l \mapsto (-1)^{kl} c' \otimes c \in C'_l \otimes C_k$  becomes after linear extension a chain isomorphism

$$\tau_\bullet : C_\bullet \otimes C'_\bullet \rightarrow C'_\bullet \otimes C_\bullet.$$

LEMMA 6.9. *Let be given a short exact sequence of free chain complexes:*

$$0 \rightarrow C'_\bullet \rightarrow C_\bullet \rightarrow C''_\bullet \rightarrow 0.$$

*Then its tensor product with any chain complex  $D_\bullet$  which is zero in negative degrees,*

$$0 \rightarrow C'_\bullet \otimes D_\bullet \rightarrow C_\bullet \otimes D_\bullet \rightarrow C''_\bullet \otimes D_\bullet \rightarrow 0,$$

*is also exact.*

PROOF. Let us write  $E_\bullet$  for the short exact sequence  $0 \rightarrow C'_\bullet \rightarrow C_\bullet \rightarrow C''_\bullet \rightarrow 0$ . For every  $r \in \mathbb{Z}$  we consider the chain complex  $D_\bullet^{\leq r}$ , the terms of which are  $D_\bullet$  in degree  $\leq r$  and zero in degree  $> r$ . We have  $D_\bullet^{\leq r} \subset D_\bullet^{\leq r+1}$  and  $D_\bullet^{\leq r} = 0$  for  $r < 0$ . We prove with induction on  $r$  that the lemma holds for  $E_\bullet \otimes D_\bullet^{\leq r}$ . This clearly suffices. We have an exact sequence

$$0 \rightarrow E_\bullet \otimes D_\bullet^{\leq r-1} \rightarrow E_\bullet \otimes D_\bullet^{\leq r} \rightarrow E_\bullet \otimes D_r \rightarrow 0.$$

By induction  $E_\bullet \otimes D_\bullet^{\leq r-1}$  is a short exact sequence of chain complexes. The same holds for  $E_\bullet \otimes D_r$  by corollary 6.7 and the fact that  $C''_k \star D_r = 0$ . The long exact homology sequence 2.2 shows that  $E_\bullet \otimes D_\bullet^{\leq r}$  is also exact.  $\square$

Let  $C_\bullet$  and  $C'_\bullet$  be chain complexes. If  $z \in Z_k(C_\bullet)$  and  $z' \in Z_l(C'_\bullet)$ , then  $d(z \otimes z') = dz \otimes z' + (-1)^k z \otimes dz' = 0$ , and so  $z \otimes z' \in Z_{k+l}(C_\bullet \otimes C'_\bullet)$ . If one of these is a boundary, say  $z = dc$ , then so is  $z \otimes z'$ , for  $d(c \otimes z') = z \otimes z'$ . This implies that we have a natural homomorphism  $H_k(C_\bullet) \otimes H_l(C'_\bullet) \rightarrow H_{k+l}(C_\bullet \otimes C'_\bullet)$ . So if we write  $H_*(C_\bullet)$  for  $\bigoplus_k H_k(C_\bullet)$ , then we have constructed the *cross product*

$$\times : H_*(C) \otimes H_*(C') \rightarrow H_*(C \otimes C').$$

We will see that this map is always injective and we will determine its cokernel.

**THEOREM 6.10 (Künneth formula).** *Let  $C_\bullet$  and  $C'_\bullet$  be chain complexes that are zero in negative degrees and assume every  $C_k$  free. Then for every  $n$  we have a natural exact sequence*

$$0 \rightarrow (H_*(C_\bullet) \otimes H_*(C'_\bullet))_n \xrightarrow{\times} H_n(C_\bullet \otimes C'_\bullet) \rightarrow (H_*(C_\bullet) \star H_*(C'_\bullet))_{n-1} \rightarrow 0.$$

*If in addition  $C'_\bullet$  is free or has only one nonzero term, then this sequence splits (and so the middle term is non-canonically isomorphic to the direct sum of its neighbors).*

**PROOF.** Since  $Z_l := Z_l(C_\bullet)$  and  $dC_l$  are subgroups of the free group  $C_l$ , they are free as well. Thus the exact sequences  $0 \rightarrow Z_l \rightarrow C_l \rightarrow dC_l \rightarrow 0$  make up a short exact sequence of free chain complexes

$$0 \rightarrow Z_\bullet \rightarrow C_\bullet \rightarrow dC_\bullet \rightarrow 0.$$

Notice that the boundary maps of the outer complexes are zero. According to Lemma 6.9 the short exact sequences of chain complexes

$$(\dagger) \quad 0 \rightarrow Z_\bullet \otimes C'_\bullet \rightarrow C_\bullet \otimes C'_\bullet \rightarrow dC_\bullet \otimes C'_\bullet \rightarrow 0$$

is also exact. The boundary map of  $Z_\bullet \otimes C'_\bullet$  is  $\pm 1 \otimes d$ . Since  $Z_\bullet$  is free it follows that

$$H_n(Z_\bullet \otimes C'_\bullet) = \bigoplus_{k+l=n} Z_k \otimes H_l(C'_\bullet)$$

Likewise we have

$$H_n(dC_\bullet \otimes C'_\bullet) = \bigoplus_{k+l=n} dC_k \otimes H_l(C'_\bullet).$$

So the long exact homology sequence of  $(\dagger)$  becomes

$$(\ddagger) \quad \dots \xrightarrow{\partial_{n+1}} \bigoplus_{k+l=n} Z'_k \otimes H_l(C_\bullet) \rightarrow H_n(C_k \otimes C'_\bullet) \rightarrow \bigoplus_{k+l=n} dC_k \otimes H_l(C'_\bullet) \xrightarrow{\partial_n} \dots$$

Now  $\partial_n$  is equal to the direct sum of the homomorphisms (up to sign)

$$i_{k-1} \otimes \mathbf{1} : dC_k \otimes H_l(C'_\bullet) \rightarrow Z_{k-1} \otimes H_l(C'_\bullet),$$

where  $i_{k-1} : dC_k \subset Z_{k-1}$  is the inclusion. Since  $dC_k$  and  $Z_{k-1}$  are free we have by 6.7 an exact sequence

$$\begin{aligned} 0 \rightarrow H_{k-1}(C_\bullet) \star H_l(C'_\bullet) &\rightarrow dC_k \otimes H_l(C'_\bullet) \xrightarrow{i_{k-1} \otimes \mathbf{1}} \\ &\rightarrow Z_{k-1} \otimes H_l(C'_\bullet) \rightarrow H_{k-1}(C_\bullet) \otimes H_l(C'_\bullet) \rightarrow 0. \end{aligned}$$

This implies

$$\begin{aligned} \text{Ker}(\partial_n) &\cong \bigoplus_{k+l=n-1} H_k(C_\bullet) \star H_l(C'_\bullet) \quad \text{and} \\ \text{Coker}(\partial_n) &\cong \bigoplus_{k+l=n-1} H_{k-1}(C_\bullet) \otimes H_l(C'_\bullet). \end{aligned}$$

If we substitute this in the sequence  $(\ddagger)$ , we find the short exact sequences of the theorem.

In order to verify that the first map is the cross product, we choose cycles  $z \in C_k$ ,  $z' \in C'_l$ . Then  $z \otimes [z'] \in Z_k \otimes H_l(C'_\bullet) \subset H_{k+l}(Z_\bullet \otimes C'_\bullet)$ . Its image under the map  $H_{k+l}(Z_\bullet \otimes C'_\bullet) \rightarrow H_{k+l}(C_\bullet \otimes C'_\bullet)$  indeed equals  $[z] \times [z']$ .

Since  $dC_k$  is free, there exists a section  $s_k : dC_k \rightarrow C_k$  of  $d : C_k \rightarrow dC_k$ . We thus find a retract  $r_k := \mathbf{1} - s_k d : C_k \rightarrow Z_k$ . Now assume that  $C'_\bullet$  is free or as a single nonzero term. Then we also have a retract  $r'_l : C'_l \rightarrow Z'_l$ . From the identity  $(r \otimes r')d(c \otimes c') = dc \otimes r'(c') \pm r(c) \otimes dc$  it follows that  $(r \otimes r')d(C_\bullet \otimes C'_\bullet) \subset dC_\bullet \otimes Z'_\bullet +$

$Z_\bullet \otimes dC'_\bullet$ . Hence  $r \otimes r'$  induces a homomorphism  $H_*(C_\bullet \otimes C'_\bullet) \rightarrow H_*(C_\bullet) \otimes H_*(C'_\bullet)$ . This is a left inverse of  $\times$ .  $\square$

**COROLLARY 6.11 (Universal coefficient theorem).** *If  $C_\bullet$  is a free chain complex that is zero in negative degrees, then for every abelian group  $M$ , there is for every  $n$  a splittable natural exact sequence*

$$0 \rightarrow H_n(C_\bullet) \otimes M \rightarrow H_n(C_\bullet \otimes M) \rightarrow H_{n-1}(C_\bullet) \star M \rightarrow 0.$$

**PROOF.** Apply the preceding theorem to the chain complex  $C'_\bullet$  with  $C'_k = 0$  for  $k \neq 0$  and  $C'_0 = M$ .  $\square$

This corollary shows that if  $K$  is a field, then we have an injective homomorphism  $H_n(C_\bullet) \otimes K \hookrightarrow H_n(C_\bullet \otimes K)$  which is even an isomorphism unless  $K$  has characteristic  $p > 0$  and  $H_{n-1}(C_\bullet)$  has  $p$ -torsion.

Here is another application of Theorem 6.10.

**COROLLARY 6.12.** *For every two abelian groups  $M$  and  $M'$ ,  $M \star M'$  is isomorphic to  $M' \star M$ .*

**PROOF.** Choose short free resolutions  $P_\bullet \rightarrow M$  and  $P'_\bullet \rightarrow M'$ . These are chain complexes whose homology is concentrated in degree zero (for which we get  $M$  resp.  $M'$ ). Applying Theorem 6.10 to  $P_\bullet$  and  $P'_\bullet$  with  $n = 1$  yields an isomorphism  $H_1(P_\bullet \otimes P'_\bullet) \cong M \star M'$ . The interchange isomorphism  $\tau$  identifies  $H_1(P_\bullet \otimes P'_\bullet)$  with  $H_1(P'_\bullet \otimes P_\bullet)$  and so  $M \star M'$  with  $M' \star M$ .  $\square$

We observed that for any group  $N$ , the functor  $\otimes N : \mathcal{A}b \rightarrow \mathcal{A}b$  is right exact. Analogously, we have

**LEMMA 6.13.** *The functor  $\text{Hom}(-, N)$  is left exact: if the sequence*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

*is exact, then so is the associated sequence*

$$\text{Hom}(A, N) \longleftarrow \text{Hom}(B, N) \longleftarrow \text{Hom}(C, N) \longleftarrow 0.$$

**PROOF.** This is easy to verify (a homomorphism  $B \rightarrow N$  factors through the quotient  $B/\alpha(A) \cong C$  if and only if it vanishes on  $\alpha(A)$ ).  $\square$

The functor  $\text{Hom}(-, N)$  need not be exact: for instance, take  $N = \mathbb{Z}$  and let the short exact sequence be  $0 \rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0$ . Then the Hom sequence becomes  $0 \rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \rightarrow 0 \rightarrow 0$ , which is evidently not exact.

Let  $C_\bullet$  be a chain complex. Then for an abelian group  $N$ ,

$$\text{Hom}(C_\bullet, N) : \cdots \leftarrow \text{Hom}(C_{k+1}, N) \leftarrow \text{Hom}(C_k, N) \leftarrow \text{Hom}(C_{k-1}, N) \leftarrow \cdots$$

is also a chain complex, except that the arrows go the other way. We can easily resolve this by giving  $\text{Hom}(C_k, N)$  degree  $-k$ . However, we prefer to give  $\text{Hom}(C_k, N)$  degree  $k$  and then regard  $\text{Hom}(C_\bullet, N)$  as a *chain cocomplex*: by definition this is a sequence of groups  $\{C^k\}_{k \in \mathbb{Z}}$  connected by homomorphisms, called *coboundaries*,  $d^k : C^k \rightarrow C^{k+1}$  which increase the index by one and are such that  $d^k d^{k-1} = 0$  for all  $k$ :

$$C^\bullet : \cdots \longrightarrow C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \longrightarrow \cdots$$

The  $k$ -th *cohomology group* of  $C^\bullet$ ,  $H^k(C^\bullet)$ , is defined as the kernel of  $d^k$  modulo the image of  $d^{k-1}$ .

From Theorem 6.5 we deduce in a similar manner as for the tensor product:

**PROPOSITION-DEFINITION 6.14.** *Let  $M$  and  $N$  be abelian groups. Then for any short free resolution  $u : P_\bullet \rightarrow M$  of  $M$ , the cokernel of the map  $\text{Hom}(P_0, N) \rightarrow \text{Hom}(P_1, N)$  induced by  $d$  is independent of the choice of the resolution. We call this cokernel the ext group of the pair  $(M, N)$  and denote it by  $\text{Ext}(M, N)$ .*

We will see that the extension group  $\text{Ext}(M, N)$  is (like  $\text{Hom}(M, N)$ ) not symmetric in  $M$  and  $N$ . As for the torsion product we find:

**COROLLARY 6.15.** *Formation of the ext group enjoys the following properties.*

**Exact sequence:** *Given a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  with  $A$  and  $B$  free, then for every abelian group  $N$  the sequence*

$$0 \rightarrow \text{Hom}(C, N) \rightarrow \text{Hom}(B, N) \rightarrow \text{Hom}(A, N) \rightarrow \text{Ext}(C, N) \rightarrow 0$$

*is exact.*

**Acyclicity of free abelian groups:** *If  $M$  is free, then  $\text{Ext}(M, N) = 0$ .*

**Functoriality:** *Two homomorphisms of abelian groups  $f : M \leftarrow M'$  and  $g : N \rightarrow N'$  determine a homomorphism  $\text{Ext}(f, g) : \text{Ext}(M, N) \rightarrow \text{Ext}(M', N')$ .*

**EXAMPLE 6.16.** We take  $M = \mathbb{Z}/(m)$  ( $m > 0$ ) and  $N = \mathbb{Z}/(n)$  ( $n \geq 0$ ). If we apply the preceding to short free resolution  $0 \rightarrow \mathbb{Z} \xrightarrow{m \times} \mathbb{Z} \rightarrow \mathbb{Z}/(m) \rightarrow 0$ , then we get the exact sequence

$$0 \leftarrow \text{Ext}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \leftarrow \mathbb{Z}/(n) \xleftarrow{m \times} \mathbb{Z}/(n) \leftarrow \text{Hom}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \leftarrow 0.$$

This shows that  $\text{Ext}(\mathbb{Z}/(m), \mathbb{Z}/(n)) \cong \mathbb{Z}/(m, n)$ . In particular, we get for  $n = 0$  (i.e.,  $N = \mathbb{Z}$ ):  $\text{Ext}(\mathbb{Z}/(m), \mathbb{Z}) \cong \mathbb{Z}/(m)$ . On the other hand,  $\text{Ext}(\mathbb{Z}, \mathbb{Z}/(m)) = 0$ . So  $\text{Ext}(N, M)$  and  $\text{Ext}(M, N)$  need not be isomorphic.

**THEOREM 6.17** (Universal coefficient theorem for cohomology). *If  $C_\bullet$  is a free chain complex that is zero in negative degrees, then for any abelian group  $N$ , there is for every  $n$  a splittable canonical exact sequence*

$$0 \rightarrow \text{Ext}(H_{n-1}(C_\bullet), N) \rightarrow H^n(\text{Hom}(C_\bullet, N)) \rightarrow \text{Hom}(H_n(C_\bullet), N) \rightarrow 0.$$

**PROOF.** The proof is similar to that of 6.11. □

So we have  $H^n(\text{Hom}(C_\bullet, \mathbb{Q})) \cong \text{Hom}(H_n(C_\bullet), \mathbb{Q})$ .

**PROBLEM 34.** Determine for every integer  $n \geq 0$  the groups  $(\mathbb{Z}/n) \otimes S^1$ ,  $(\mathbb{Z}/n) \star S^1$ ,  $\text{Hom}(\mathbb{Z}/n, S^1)$  and  $\text{Ext}(\mathbb{Z}/n, S^1)$ . Same question when  $S^1$  is replaced by  $\mathbb{Q}/\mathbb{Z}$ .

**PROBLEM 35.** Let  $C_\bullet$  be a free chain complex of abelian groups whose total homology  $H_*(C_\bullet)$  is finitely generated as an abelian group. Prove that for every field  $K$ , the Euler characteristic with coefficients in  $K$ ,  $\sum_k (-1)^k \dim_K H_k(C_\bullet \otimes K)$ , is equal to the Euler characteristic  $\sum_k (-1)^k \text{rank } H_k(C_\bullet)$ . (Hint: the essential case is when  $K$  is a prime field:  $K = \mathbb{Q}$  or  $K = \mathbb{F}_p$ .)

**PROBLEM 36.** Let  $M$  be an abelian group and  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  a short exact sequence.

(a) Let  $u' : P'_\bullet \rightarrow N'$  and  $u'' : P''_\bullet \rightarrow N''$  be a short free resolutions. Prove that we have a short free resolution  $u : P_\bullet \rightarrow N$  which extends  $u'$  and has  $u''$  as quotient.

(b) Show that we have an exact sequence

$$0 \rightarrow M \star N' \rightarrow M \star N \rightarrow M \star N'' \rightarrow M \otimes N' \rightarrow M \otimes N \rightarrow M \otimes N'' \rightarrow 0.$$

(c) Show that we have an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}(M, N') \rightarrow \text{Ext}(M, N) \rightarrow \text{Ext}(M, N'') \rightarrow \\ \rightarrow \text{Hom}(M, N') \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}(M, N'') \rightarrow 0. \end{aligned}$$

## 7. Applications to Topology

Let  $(X, A)$  be a space pair and  $M$  an abelian group. Recall that the group of  $C_k(X, A)$  is the free abelian group generated by the singular  $k$ -simplices that map to  $X$  but not to  $A$ . We define the group of *singular  $k$ -chains of  $(X, A)$  with coefficients in  $M$*  is the tensor product  $C_k(X, A; M) := C_k(X, A) \otimes_{\mathbb{Z}} M$ . Since  $C_k(X, A) = C_k(X)/C_k(A)$  this is also equal to  $(C_k(X) \otimes M)/(C_k(A) \otimes M)$ .

We have a chain complex  $C_{\bullet}(X, A; M)$  whose homology we shall denote by  $H_*(X, A; M)$ . It is called the *homology of  $(X, A)$  with coefficients in  $M$* .

**THEOREM 7.1.** *For every  $k \in \mathbb{Z}$  we have splittable natural exact sequence*

$$0 \rightarrow H_k(X, A) \otimes M \rightarrow H_k(X, A; M) \rightarrow H_{k-1}(X, A) \star M \rightarrow 0.$$

Furthermore, the fundamental properties I-IV for homology also hold for homology with coefficients in  $M$ , except that the dimension property reads  $H_0(\{p\}; M) = M$  and  $H_k(\{p\}; M) = 0$  for  $k \neq 0$ .

**PROOF.** The first assertion is immediate from 6.11. The rest is straightforward. For instance, the long exact sequence for a space triple  $(X, Y, Z)$  is obtained from the exact sequence of complexes

$$0 \rightarrow C_{\bullet}(Y, Z) \otimes M \rightarrow C_{\bullet}(X, Z) \otimes M \rightarrow C_{\bullet}(X, Y) \otimes M \rightarrow 0.$$

(Since we tensored short exact sequences of free groups with  $M$ , these sequences remain exact.)  $\square$

The group of *singular  $k$ -cochains of  $(X, A)$  with values in  $M$*  is the group of homomorphisms  $\text{Hom}(C_k(X, A), M)$ . Notice that these are the homomorphisms  $C_k(X) \rightarrow M$  that vanish on  $C_k(A)$ . We denote this group by  $C^k(X, A; M)$ . We have cochain complex  $C^{\bullet}(X, A; M)$  whose cohomology is called the *cohomology of  $(X, A)$  with values in  $M$* ; it is denoted by  $H^k(X, A; M)$ . We deduce from 6.17:

**THEOREM 7.2.** *For every  $k \in \mathbb{Z}$  we have splittable natural exact sequence*

$$0 \rightarrow \text{Ext}(H_{k-1}(X, A), M) \rightarrow H^k(X, A; M) \rightarrow \text{Hom}(H_k(X, A), M) \rightarrow 0.$$

Furthermore, the fundamental properties I-IV for homology also hold for cohomology with values in  $M$ , except that the dimension property reads  $H^0(\{p\}; M) = M$  and  $H^k(\{p\}; M) = 0$  for  $k \neq 0$ , and the arrows go the other way.

For example, a triple  $(X, Y, Z)$  gives rise to a long exact sequence

$$\dots \rightarrow H^k(X, Y) \rightarrow H^k(X, Z) \rightarrow H^k(Y, Z) \xrightarrow{\delta^k} H^{k+1}(X, Y) \rightarrow \dots$$

**EXAMPLE 7.3.** Let us compute  $H_k(P^n; \mathbb{Z}/(2))$  and  $H^k(P^n; \mathbb{Z}/(2))$ . We get from 4.20 that

$$H_k(P^n) \otimes \mathbb{Z}/(2) \cong \begin{cases} \mathbb{Z}/(2) & \text{if } k = 0 \text{ or } k \text{ odd and } 1 \leq k \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

$$H_{k-1}(P^n) \star \mathbb{Z}/(2) \cong \begin{cases} \mathbb{Z}/(2) & \text{if } k \text{ even and } 2 \leq k \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that  $H_k(P^n; \mathbb{Z}/(2)) \cong \mathbb{Z}/(2)$  for  $k = 0, 1, \dots, n$  and is zero otherwise. We similarly find that  $H^k(P^n; \mathbb{Z}/(2)) \cong \mathbb{Z}/(2)$  for  $k = 0, 1, \dots, n$  and is zero otherwise.

Theorem 7.1 gives us for  $M = \mathbb{Z}$  a map  $H^k(X, A) \rightarrow \text{Hom}(H_k(X, A), \mathbb{Z})$ . The associated bilinear map

$$H^k(X, A) \times H_k(X, A) \rightarrow \mathbb{Z}, \quad (\xi, u) \mapsto \langle \xi, u \rangle$$

is usually referred to as the *duality pairing between homology and cohomology*.

REMARK 7.4. For  $M = \mathbb{R}$ , Theorem 7.1 yields the isomorphism  $H^k(X, A; \mathbb{R}) \cong \text{Hom}(H_k(X, A), \mathbb{R})$ . When  $X$  is a differentiable manifold, then we may limit ourselves to singular simplices  $\Delta^k \rightarrow X$  that are differentiable. These define a subcomplex  $C_\bullet^\infty(X) \subset C_\bullet(X)$  and one can show that this inclusion induces an isomorphism on homology. A differential  $k$ -form  $\alpha$  on  $X$  can be integrated over a differentiable  $k$ -chain  $a$  so that we get  $\int_a \alpha \in \mathbb{R}$ . If  $a$  is a cycle and  $\alpha$  is closed, then a version of Stokes' theorem shows that this integral is zero if  $a$  is a boundary or  $\alpha$  is exact. Thus we obtain a linear map from the  $k$ th De Rham cohomology space  $H_{DR}^k(X)$  to  $\text{Hom}(H_k(X), \mathbb{R})$ . A fundamental theorem due to De Rham states that this map is an isomorphism. In view of the preceding, we may also state this as: *integration identifies the De Rham cohomology of a differentiable manifold with its cohomology with coefficients in  $\mathbb{R}$* . Via this identification the cup product that we will define below is given by the wedge product of differentiable forms.

For a space  $X$  we define a homomorphism  $\epsilon_* : C_\bullet(X) \rightarrow \mathbb{Z}$  by letting it be zero in every degree  $\neq 0$  and assign 1 to every singular 0-simplex. We may regard this as a chain map if we identify  $\mathbb{Z}$  with the chain complex whose only nonzero term is  $\mathbb{Z}$  placed in degree zero. This homomorphism is called the *co-augmentation*. Its dual gives the *augmentation*

$$\epsilon^* : \mathbb{Z} \rightarrow C^\bullet(X).$$

The associated homomorphisms on (co)homology are denoted likewise:

$$\epsilon_* : H_*(X) \rightarrow \mathbb{Z}, \quad \epsilon^* : \mathbb{Z} \rightarrow H^*(X).$$

Note that they only live in degree zero. If  $X$  is nonempty, then these homomorphisms are surjective resp. injective. In the last case we may identify  $\mathbb{Z}$  with its image in  $C^0(X)$  so that  $1 \in \mathbb{Z}$  is represented by the 0-cocycle which assigns to every singular 0-simplex of  $X$  the value 1.

Let us define  $D^k \in (C_\bullet(\Delta^k) \otimes C_\bullet(\Delta^k))_k$  by

$$D^k = \sum_{i=0}^k \langle e_0, \dots, e_i \rangle \otimes \langle e_i, \dots, e_k \rangle.$$

In the standard manner this produces for every  $X$  a homomorphism

$$\delta_k : C_k(X) \rightarrow (C_\bullet(X) \otimes C_\bullet(X))_k, \quad \sigma \mapsto (\sigma_* \otimes \sigma_*)(D^k).$$

The following properties are verified in a straightforward manner.

LEMMA 7.5. *The  $\delta_k$ 's define a chain map  $\delta_\bullet : C_\bullet(X) \rightarrow C_\bullet(X) \otimes C_\bullet(X)$  which enjoys the following properties:*

**Functoriality:** For any map  $f : X \rightarrow Y$ , we have  $\delta_\bullet f_* = (f_* \otimes f_*)\delta_\bullet$ ,

**Co-associativity:** we have

$$\delta_\bullet(\mathbf{1} \otimes \delta_\bullet) = \delta_\bullet(\delta_\bullet \otimes \mathbf{1}) : C_\bullet(X) \rightarrow C_\bullet(X) \otimes C_\bullet(X) \otimes C_\bullet(X),$$

**Cocommutativity:** if  $\tau : C_\bullet(X) \otimes C_\bullet(X) \rightarrow C_\bullet(X) \otimes C_\bullet(X)$  is defined by  $u \otimes v \in C_k(X) \otimes C_l(X) \mapsto (-1)^{kl}v \otimes u$ , then  $\delta_\bullet \tau = \delta_\bullet$ ,

**Co-unit:** if  $X \neq \emptyset$ , then  $(\mathbf{1} \otimes \epsilon_*)\delta_\bullet = (\epsilon_* \otimes \mathbf{1})\delta_\bullet = \mathbf{1} : C_\bullet(X) \rightarrow C_\bullet(X)$ .

The chain map  $\delta_\bullet$  also yields functorial homomorphisms on cohomology:

$$H^n(\delta_\bullet) : H^n(C_\bullet(X) \otimes C_\bullet(X)) \rightarrow H^n(C_\bullet(X)) = H^n(X; R).$$

If we compose it with the cross product, we get a homomorphism

$$H^k(X) \otimes H^l(X) \rightarrow H^{k+l}(X),$$

It is called the *cup product* because the image of  $u \otimes v$  is denoted  $u \cup v$ . From Lemma 7.5 we get that the cup product turns the total cohomology into graded commutative ring in such a manner that maps induce ring homomorphisms:

**PROPOSITION 7.6.** *The cup product is associative, graded commutative ( $u \cup v = (-1)^{kl}v \cup u$  when  $u \in H^k(X)$  and  $v \in H^l(X)$ ) and for nonempty  $X$ ,  $1 \in H^0(X)$  is unit:  $1 \cup u = u \cup 1 = u$ . Moreover, if  $f : Y \rightarrow X$  is a map, then the induced map on cohomology commutes with the cup product.*

**EXAMPLE 7.7.** The universal coefficient theorem for cohomology implies that  $H^k(P_n(\mathbb{C}))$  is equal to  $\mathbb{Z}$  for  $k = 0, 2, 4, \dots, 2n$  and zero otherwise. It can be proved that an additive generator  $\alpha \in H^2(P_n(\mathbb{C}))$  is also a multiplicative generator of  $H^*(P_n(\mathbb{C}))$ :  $H^*(P_n(\mathbb{C})) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$ .

A similar statement holds for  $H^*(P_n, \mathbb{Z}/(2))$ .

The duality pairing  $\langle \cdot, \cdot \rangle$  between homology and cohomology can be extended to arbitrary degrees: we have a chain homomorphism

$$C^\bullet(X) \otimes C_\bullet(X) \xrightarrow{\mathbf{1} \otimes \delta_\bullet} C^\bullet(X) \otimes C_\bullet(X) \otimes C_\bullet(X) \xrightarrow{\langle \cdot, \cdot \rangle \otimes \mathbf{1}} C_\bullet(X).$$

which induces for every  $k$  and  $l$  a homomorphism

$$H^k(X) \otimes H_l(X) \rightarrow H_{l-k}(X)$$

which we call the *cap product* and denote  $\alpha \otimes a \mapsto \alpha \cap a$ . A straightforward check shows that the total homology  $H_*(X)$  becomes for nonempty  $X$ , a module over its cohomology ring  $H^*(X)$ . For instance

$$\xi \cap (\eta \cap u) = (\xi \cup \eta) \cap u.$$

Similarly, the chain homomorphism

$$C^\bullet(X) \otimes C_\bullet(X) \xrightarrow{\delta_\bullet^* \otimes \mathbf{1}} C^\bullet(X) \otimes C^\bullet(X) \otimes C_\bullet(X) \xrightarrow{\mathbf{1} \otimes \langle \cdot, \cdot \rangle} C^\bullet(X).$$

induces for every  $k$  and  $l$  a homomorphism

$$H^k(X) \otimes H_l(X) \rightarrow H^{k-l}(X),$$

called the *slant product* because it is denoted  $\xi \otimes u \mapsto \xi/u$ . As the cup product, it has a number of compatibility properties which we will not bother to list in an exhaustive manner. For instance, the definition shows that  $(\xi \cup \eta)/u = \xi \cup (\eta/u)$ .

For evident reasons the cap (resp. slant) product is zero for  $k > l$  (resp.  $k < l$ ). For  $k = l$  either product becomes the natural duality pairing.



We can do the preceding also for pairs. Let  $A$  and  $B$  be open subsets of  $X$ . The homomorphism  $\delta_\bullet : C_\bullet(X) \rightarrow C_\bullet(X) \otimes C_\bullet(X)$  maps  $C_\bullet(A)$  to  $C_\bullet(A) \otimes C_\bullet(A) \subset C_\bullet(A) \otimes C_\bullet(X)$  and  $C_\bullet(B)$  to  $C_\bullet(B) \otimes C_\bullet(B) \subset C_\bullet(X) \otimes C_\bullet(B)$ . So  $\delta_\bullet$  induces a chain map

$$C_\bullet(X)/(C_\bullet(A) + C_\bullet(B)) \rightarrow C_\bullet(X, A) \otimes C_\bullet(X, B).$$

By Proposition 2.12, the inclusion  $C_\bullet(A) + C_\bullet(B) \rightarrow C_\bullet(A \cup B)$  induces an isomorphism on homology. Hence the same is true for the surjection

$$C_\bullet(X)/(C_\bullet(A) + C_\bullet(B)) \rightarrow C_\bullet(X)/C_\bullet(A \cup B)$$

(see problem 13). We thus find

**PROPOSITION 7.8.** *For  $A, B$  open in  $X$  we have a cup product*

$$H^k(X, A) \otimes H^l(X, B) \rightarrow H^{k+l}(X, A \cup B), \quad \xi \otimes \eta \mapsto \xi \cup \eta$$

*which is functorial, associative and graded commutative. We also have a functorial cap product and slant product*

$$H^k(X, A) \otimes H_l(X, A \cup B) \rightarrow H_{l-k}(X, B), \quad \xi \otimes u \mapsto \xi \cap u$$

$$H^k(X, A \cup B) \otimes H_l(X, B) \rightarrow H^{k-l}(X, A), \quad \xi \otimes u \mapsto \xi / u.$$

**PROBLEM 37.** Compute the homology and cohomology of the Klein bottle with  $\mathbb{F}_2$ -coefficients.

**PROBLEM 38.** Prove the following functoriality property: Let  $f : X \rightarrow Y$  be a map of spaces,  $u \in H_*(X)$  and  $\eta \in H^*(Y)$ . Show that  $f_*(f^*\eta \cap u) = \eta \cap f_*u$ .