Extra exercises Introduction to Algebraic Topology (Fall 2011)
Part of the 8 EC version

Exercise 1. Let S^2 be the 2-sphere and let $D_1, \ldots, D_n$ be n small open discs on S^2 with disjoint boundaries. Let X^+, X^- be two copies of $S^2 \setminus (D_1 \cup \ldots \cup D_n)$ and let X be the space obtained by identifying, for each $i = 1, \ldots, n$, the boundary of D_i on X^+ with the boundary of D_i on X^-, using the identity map. Thus, X is a "sphere with $n-1$ handles". Compute the homology groups of X.

Exercise 2. In this exercise we prove that an even-dimensional sphere cannot be given the structure of a topological group. Given a group G acting as a group of homeomorphisms of a space X, we say that G acts *freely* if the only element from G which has any fixed points is the identity element. Let g, h be two elements, unequal to the identity element, from a group G acting freely on S^n, where $n > 0$ is even.
(a) Prove that both g and h have degree -1.
(b) Prove that gh is the identity element.
(c) Conclude that G is either $\mathbb{Z}/2\mathbb{Z}$ or the trivial group.
(d) Prove that S^n is not a topological group.

Exercise 3. Let $n \in \mathbb{Z}_{>0}$. Prove that every continuous map $S^n \to S^n$ is homotopic to one that has a fixed point.

Exercise 4. Let $\mathcal{A}$ be an abelian category in which each short exact sequence splits (e.g., the category of vector spaces over a field k). Such an abelian category is called *semisimple.*
Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor to an abelian category $\mathcal{B}$. Prove that the right derived functors of F exist, and that they are zero in positive degree.

For each integer $n \geq 0$ we let:

$$g(n) := \min\{N \in \mathbb{Z} \,|\, \text{there exists a topological space } X \text{ with } \#X = N \text{ and } \mathrm{H}^n(X, \mathbb{Z}_X) \neq 0\}\,.$$

Here $\mathbb{Z}_X$ denotes the constant sheaf with values in $\mathbb{Z}$ on X. The purpose of the exercises below is to prove a result of S.J. Edixhoven and R. Noot, saying that $g(0) = 1$, and $g(n) = 2n + 2$ if $n > 0$.

Exercise 5. Let X be a finite set. Prove that giving a topology on X is equivalent with giving a pre-ordering on X. A pre-ordering on X is a relation which is reflexive and transitive.
Hint: if X is a topological space, then the relation $x \geq y \Leftrightarrow y \in \overline{\{x\}}$ defines a pre-ordering on X.

Exercise 6. Let X be a finite topological space given by a pre-ordering $\geq$. Put an equivalence relation $\sim$ on X by letting $x \sim y \Leftrightarrow x \geq y$ and $y \geq x$.
(i) Show that the quotient space $X/\!\sim$ is naturally a partially ordered set.
(ii) Show that the categories $Op\,X$ and $Op\,X/\!\sim$ of open sets of X resp. $X/\!\sim$ are isomorphic. That is, there are natural functors $F\colon Op\,X \to Op\,X/\!\sim$ and $G\colon Op\,X/\!\sim\, \to Op\,X$ whose compositions give the identity functor on $Op\,X$ resp. $Op\,X/\!\sim$.

Exercise 7. A topological space X is called *irreducible* if $X \neq \varnothing$ and if an equality $X = V \cup W$ with V and W closed in X implies that at least one of V, W is equal to X.
Prove that $\mathbb{Z}_X$ is flasque on each irreducible space. Hint: prove that a non-empty open subset of an irreducible space is irreducible.

Now let $n > 0$ be an integer and let X_0 be a topological space with $\#X_0 = g(n)$ and $\mathrm{H}^n(X_0, \mathbb{Z}) \neq 0$.

Exercise 8. (i) Show that X_0 is a partially ordered set.
(ii) Show that X_0 is not irreducible.

Exercise 9. Let $x_1, \ldots, x_r$ be the maximal points of X_0 for the partial ordering on X_0.
(i) Show that $r \geq 2$.
Let $X_1 = \overline{\{x_1\}}$, $X_2 = \overline{\{x_2, \ldots, x_r\}}$.
(ii) Use the Mayer-Vietoris exact sequence (see sheet 2, Exercise 7) to show that:

$$\mathrm{H}^{n-1}(X_1 \cap X_2, \mathbb{Z}) \neq 0\,.$$

(iii) Deduce that $g(n) \geq g(n-1) + 2$ for $n > 0$.

If Y is a finite partially ordered set we define $\mathbb{S}Y$ to be the finite set $Y \sqcup \{z_1, z_2\}$ with partial ordering given by declaring z_1, z_2 to be maximal, and $\overline{\{z_i\}} = Y \sqcup \{z_i\}$ for $i = 1, 2$.

Exercise 10. Use the Mayer-Vietoris exact sequence to show that $\mathrm{H}^{n+1}(\mathbb{S}X_0, \mathbb{Z}) \neq 0$. Deduce from this that $g(n+1) \leq g(n) + 2$ for $n > 0$.

Exercise 11. The previous two exercises show that $g(n) = g(n-1) + 2$ for $n > 1$. The following two items finish the proof of the result of Edixhoven and Noot.
(i) Show that $g(0) = 1$.
(ii) Show that $g(1) = 4$.
Hint: check that all spaces X with $\#X \leq 3$ have $\mathrm{H}^1(X, \mathbb{Z}) = 0$, and that $\mathrm{H}^1(\mathbb{S}X, \mathbb{Z}) \neq 0$ for $X = S^0$, the discrete space with 2 elements.

The exercises above show that for $X = S^0$ the discrete space with 2 elements and for $n > 0$, the topological space $\mathbb{S}^n X := \mathbb{S} \cdots \mathbb{S}X$ (the operation $\mathbb{S}$ applied n times, starting with X) has $2n + 2$ elements and has $\mathrm{H}^n(\mathbb{S}^n X, \mathbb{Z}) \neq 0$. In fact one can prove:

$$\mathrm{H}^k(\mathbb{S}^n X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{else.} \end{cases}$$

One calls $\mathbb{S}^n X$ a *finite n-sphere.*