Exercises Reading course Algebraic Topology - Covering spaces II, Fall 2013

Let X be a topological space.

Exercise 1. Let $p\colon Y \to X$ and $q\colon Z \to X$ be covers and let $r\colon Y \to Z$ be a map of covers of X. Let $x \in X$ be a base point. Prove that

$$r(\alpha \cdot y) = \alpha \cdot r(y)$$

for all $\alpha \in \pi_1(X,x)$ and all $y \in p^{-1}(x)$. Here the $\cdot$ stands for the monodromy action of $\pi_1(X,x)$.

Exercise 2. Let $p\colon Y \to X$ be a cover, with Y path-connected. Let $x \in X$ and $y, y' \in Y$ be such that $p(y) = p(y') = x$.
(i) Prove that the stabilizers Stab_y resp. $\mathrm{Stab}_{y'}$ of y resp. y' in $\pi_1(X,x)$ under the monodromy action are conjugate subgroups of $\pi_1(X,x)$.
(ii) Show that Stab_y is naturally isomorphic to $\pi_1(Y,y)$.

Exercise 3. Let $\mathbf{C}$ be a category and $F\colon \mathbf{C} \to \mathrm{Sets}$ be a covariant functor. Assume F is representable, represented by an object $\tilde{X}$ of $\mathbf{C}$. Identify a canonical element $\tilde{x}$ of $F(\tilde{X})$ and show that every morphism $\pi \in \mathrm{Hom}(\tilde{X}, Y)$ is determined by the element $F(\pi)(\tilde{x})$ of $F(Y)$.

Exercise 4. Let G be a group and S a transitive left G-set. Identify S with the coset space G/H for some subgroup H of G. Let $N_G(H)$ be the subset

$$N_G(H) = \{g \in G \,:\, gHg^{-1} = H\}$$

of G.
(i) Show that $N_G(H)$ is a subgroup of G, containing H as a normal subgroup.
(ii) Establish a group isomorphism

$$\phi\colon N_G(H)/H \xrightarrow{\sim} \mathrm{Aut}_G(S)^{op}\,.$$

Of course, this includes showing that your ϕ is a well-defined group homomorphism, and indeed is bijective.
(iii) Assume $p\colon (Y,y) \to (X,x)$ is a path-connected cover with X locally simply connected. Let $\pi_1(X,x)$ be the fundamental group of X and $\pi_1(Y,y) \subset \pi_1(X,x)$ be the fundamental group of (Y,y) (see Exercise 2). Prove that $\mathrm{Aut}(Y/X)$ is isomorphic to $N_{\pi_1(X,x)}(\pi_1(Y,y))/\pi_1(Y,y)$.

Exercise 5. Let S_3 be the symmetric group on three letters.
(i) Classify the transitive S_3-sets up to isomorphism.
(ii) Describe the Hom-sets between each pair of transitive S_3-sets.

Exercise 6. Let X be the plane $\mathbb{R}^2$ with 2 points removed.
(i) What is the fundamental group of X?
(ii) Show that there is a connected Galois cover $p\colon Y \to X$ with automorphism group isomorphic to S_3.