Hyperellipticity of curves

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In this project we will work with projective curves defined over number fields and their reductions modulo different primes.

Definition 0.1. *The genus of a projective curve C/k defined over a perfect field, is the dimension of the $k-$vector space of regular differentials $\Omega^1(C)$, we denote it by $g(C)$.*

For curves defined over the complex numbers, the genus is simply the number of "holes".

A smooth plane curve defined by a homogenous polynomial $F(x:y:z)=0$ of degree d has genus equal to $\frac{(d-1)(d-2)}{2}$.

Definition 0.2. *A curve is said to be hyperelliptic if it can be writen as*

$$C: y^2 = f(x),$$

where $f(x)$ is a polynomial in x without multiple roots. Otherwise, we say that it is non-hyperelliptic.

A hyperelliptic curve defined by a degree d polynomial $f(x)$ has genus equal to $\lfloor \frac{d-1}{2} \rfloor$.

All the genus 1 and 2 curves are hyperelliptic. The first examples of non-hyperelliptic curves appear for genus 3, and all non-hyperelliptic genus 3 curves can be written as plane quartic curves, that is, as $F(x:y:z)=0$ for a homogenous degree 4 polynomial, like $x^4+y^4+z^4=0$ or $x^3y+y^3z+z^3x=0$.

When we consider these curves defined over a finite field, that is, we consider the reduction of the curves modulo a prime number, it may happen that they become hyperelliptic.

There are two special numbers attached to the equation of a plane quartic curve: χ_{18} and Σ_{140}. The first one is different from zero, and if the polynomial $F(x:y:z)$ defining the quartic curve has integer coefficient, then these numbers are also integers. For the primes numbers p that divide χ_{18} but not Σ_{140}, we have that the reduction of the curve modulo p is hyperelliptic.

The goal of these project will be to find explicit examples of plane quartic curves whose reduction modulo certain primes become hyperelliptic.

A second goal will be to find examples of higher genus.