Differentials of zero-dimensional algebras

This project is suitable for a student who likes commutative algebra and is also interested in explicit computational work, possibly using a computer algebra system.

The "zero-dimensional algebras" in the title of the project are commutative rings A that are equipped with a ring homomorphism from some field K to A and that have the property that the dimension of A, when viewed as a K-vector space, is finite. Examples: A can be any field extension of finite degree over K; or $A = K[X]/(g)$ for some non-zero $g \in K[X]$; or $A = K \times K \times \ldots \times K$ with component-wise ring operations.

The first thing to be done is understanding the structure of such rings as products of so-called "local Artin rings". A special role will be played by the subset A_{sep} of A consisting of those $a \in A$ for which there exists $f \in K[X]$ with $f(a) = 0$ and $\gcd(f, f') = 1$, where f' denotes the derivative of f. This subset turns out to be a subring of A.

We are now interested in "differentiating" the elements of A, that is, in studying A-modules M and K-linear maps $d\colon A \to M$ with the property that for all $a, b \in A$ one has $d(ab) = a \cdot d(b) + b \cdot d(a)$. Such maps are understood through the so-called module of *Kähler differentials* of A over K, and these form the second subject to be studied by the student.

It is not difficult to show that, when K, A, A_{sep}, M, d are as above, one has the inclusion $A_{\mathrm{sep}} \subset \ker d$. Can one, for any K and A, construct M and d in such a way that one has the equality $A_{\mathrm{sep}} = \ker d$? It turns out that in the case K has non-zero characteristic, it is easy to construct examples of rings A for which this is wrong. Assume therefore now that K has characteristic zero. In that case, counterexamples are harder to construct, but they also exist. What is, among all counterexamples, the least value of $\dim_K A$? The least known value equals 10, and there is apparently a proof (to be verified by the student!) that there is no counterexample with $\dim_K A \leq 7$. Do $\dim_K A = 8$ or $\dim_K A = 9$ occur? It is hoped that the student will be able to answer this question, with a combination of theoretical and computational techniques.

Supervisors: Maarten Derickx and Hendrik Lenstra