Project: Distribution of Legendre character sums

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The aim of this project is to study the Legendre character sums, which are defined as follows. Let $\mathbb{F}_q$ be a finite field of odd characteristic, we denote by $\mu : \mathbb{F}_q^\times \to \{\pm 1\}$ the quadratic character defined by: $\mu(x) = 1$ if $x \in \mathbb{F}_q^\times$ is a square in $\mathbb{F}_q^\times$ and $\mu(x) = -1$ otherwise.

Definition 0.1. — For any parameter $b \in \mathbb{F}_q^\times$ and any (multiplicative) character $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$, define the associated Legendre character sum by:

$$S_q(\chi, b) := \sum_{x \in \mathbb{F}_q^\times} \chi(x) \cdot \mu(x^2 + 2bx + 1).$$

These sums appear when one tries to count the number of solutions (x, y) in $\mathbb{F}_q^2$ of the equation

$$y^2 = x^{2(q-1)} + 2bx^{q-1} + 1.$$

That is, the Legendre sums appear as zeroes of the zeta function of the hyperelliptic curve $C/\mathbb{F}_q$ whose affine part has equation $y^2 = x^{2(q-1)} + 2bx^{q-1} + 1$. They were originally defined to provide a finite field analogue of the classical Legendre orthogonal polynomials (hence their name), they also show up in L-functions of some elliptic curves, etc. Contrarily to other types of character sums (such as Gauss, Jacobi or Kloosterman sums, ...), they have not been studied a lot.

By definition, the Legendre sums satisfy a trivial bound: $|S_q(\chi, b)| \leq q$. But the connection with zeta functions allows to prove a much better bound:

Theorem 0.2. — *For any $b \in \mathbb{F}_q \smallsetminus \{\pm 1\}$ and any nontrivial character χ, one has $S(\chi, b) \in \mathbb{R}$ and*

$$|S_q(\chi, b)| \leq 2\sqrt{q}.$$

To each b and χ as in the theorem, one associates an "angle" $\theta_{b,\chi} \in [0, \pi]$ such that

$$S_q(\chi, b) = 2\sqrt{q} \cdot \cos(\theta_{b,\chi}).$$

The question is then to study how the angles $\{\theta_{b,\chi}\}_\chi$ are distributed in $[0, \pi]$ when b is fixed and χ varies among all possible (nontrivial) characters. More precisely, for any finite field $\mathbb{F}_q$, the procedure above gives $q - 1$ angles $\theta_{b,\chi} \in [0, \pi]$, so that, when $q \to \infty$, we get more and more angles. Do they spread out evenly in the interval $[0, \pi]$? or do they stay in some smaller interval $I \subset [0, \pi]$?

A good start to this project would be to reconstruct the proof of Theorem 0.2. Setting up some computer experiments could be of use, and would provide experimental evidence as to the distribution of the angles. Then, it seems possible to adapt to the Legendre sums some of the techniques used to study the angles of other character sums. Depending on the taste of the student, this project can be adapted to have a somewhat different direction: understanding the connection of these sums with the classical Legendre polynomials, clarifying their appearance in zeta functions, etc. Some familiarity with character sums over finite fields and zeta functions of curves is desirable.