Project: Size of solutions of some diophantine cubic equations

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Here are two arithmetical problems taken from the book "*Canterbury puzzles; Amusements in mathematics*" by Henry E. Dudeney (published in the 1920's):

Problem 1 – A merchant has some silver: he always keeps it under the form of two cubes. He had $16\,\mathrm{cm}^3$ of silver, that he had divided in 2 cubes of side-length 2 cm. After a transaction, he has earned one more cube centimetre of silver.

Can you give the dimensions (in rational numbers) of two cubes whose volumes add to $17\,\mathrm{cm}^3$? In more mathematical terms, the problem is to find $x, y \in \mathbb{Q}$ with $x, y > 0$ and $x^3 + y^3 = 17$.

Problem 2 – A doctor has two full spherical bottles containing a drug: one sphere has diameter 1 cm and the other 2 cm. She wants to transfer the contents of these spheres into two other bottles which should also be spherical.

She asks whether you can find the diameters (in rational numbers) of two spherical bottles which can together contain the same volume of drug? In more mathematical terms, the problem is now to find $x, y \in \mathbb{Q}$ with $x, y > 0$ and $x^3 + y^3 = 1^3 + 2^3 = 9$ (and x, y different from $1, 2$).

One cannot easily find solutions to these two problems by trial-and-error. For **Problem 1**, the only pair $(x, y) \in \mathbb{Q}^2$ which you can find is $x = 18/7$, $y = -1/7$... but that does not answer the question because $y < 0$. At the end of the book by Dudeney, you can find the following solutions: the smallest pairs (x_1, y_1) (resp. (x_2, y_2)) satisfying the requirement of **Problem 1** (resp. **Problem 2**) are

$$(x_1, y_1) = \left(\frac{104940}{40831}, \frac{11663}{40831}\right), \quad (x_2, y_2) = \left(\frac{415280564497}{348671682660}, \frac{676702467503}{348671682660}\right) \;!!$$

Dudeney does not give any hint as to how he found these solutions... Certainly he cannot have found them by pure chance. The first aim of the project is to recover Dudeney's solutions, to explain how he found them, and to try to "explain" why the numbers involved in them are so huge.

Both problems can be seen as special cases of the following: we are given a cube-free integer $n \geq 2$ and we try to find elements[(1)] in the set $E(n) := \{(x, y) \in \mathbb{Q}^2 \ : \ x^3 + y^3 = n\} \subset \mathbb{Q}^2 \subset \mathbb{R}^2$. An important step of the project will be to prove:

Fact – If the set $E(n)$ is nonempty then it is infinite. In that case, $E(n)$ contains infinitely many pairs (x, y) with $x, y > 0$.

This fact was certainly known to Fermat in the XVII$^{\text{th}}$ century, but one can also give a nice more modern proof based on the theory of *elliptic curves*. You will essentially rely on two ingredients:

1. If $E(n)$ is nonempty, it can be equipped with the structure of an abelian group. The group law is constructed in a "geometric manner".
2. One can measure the "size" of an element $P \in E(n)$ by introducing its *height* $h(P)$. The height function $h : E(n) \to \mathbb{R}_+$ has nice properties with respect to the group law on $E(n)$.

There are then several directions in which the project can go, depending on the student's taste. As an example, one can ask the following question: for a large cube-free integer $n \geq 2$ such that $E(n)$ is nonempty, it is expected that the "size" of the "smallest" $P \in E(n)$ has the same order of magnitude as $\log n$. It seems difficult to prove this in general but it should be doable to show that this holds for infinitely many such n.

Suggested bibliography –

– J. Cassels, *Lectures on elliptic curves*, Cambridge University Press, 1991.
– M. Hindry, *Arithmetics*, Springer, 2011.
– K. Ireland & M. Rosen, *A classical introduction to modern number theory*, Springer, 1990.
– J. Silverman & J. Tate, *Rational points on elliptic curves*, Springer, 2015.

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(1) The elements of $E(n)$ should also satisfy a positivity condition, which we temporarily ignore.