Exercise sheet I - 'Jacobians and theta functions' - Fall 2010

Exercise 1. Let $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ be a non-empty open subset and $u \in U$ a point. Recall that the real tangent space $T_{U,u}$ is naturally a $\mathbb{C}$-vector space if we put $\frac{\partial}{\partial x_\alpha} = i\frac{\partial}{\partial y_\alpha}$.
(i) Show that we have a natural direct sum decomposition as $\mathbb{C}$-vector spaces

$$\mathrm{Hom}_{\mathbb{R}}(T_{U,u}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(T_{U,u}, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(T_{U,u}, \mathbb{C})}\,.$$

(ii) Show that $\overline{\mathrm{Hom}_{\mathbb{C}}(T_{U,u}, \mathbb{C})}$ can be identified with the set of f in $\mathrm{Hom}_{\mathbb{R}}(T_{U,u}, \mathbb{C})$ which are anti-linear, i.e. that satisfy $f(iz) = -if(z)$ for all $z \in T_{U,u}$.
Let $z_1, \ldots, z_n$ be the coordinate functions on $\mathbb{C}^n$.
(iii) Show that $(\mathrm{d}z_1, \ldots, \mathrm{d}z_n)$ is a basis of $\mathrm{Hom}_{\mathbb{C}}(T_{U,u}, \mathbb{C})$, and that $(\mathrm{d}\overline{z}_1, \ldots, \mathrm{d}\overline{z}_n)$ is a basis of $\overline{\mathrm{Hom}_{\mathbb{C}}(T_{U,u}, \mathbb{C})}$.

Exercise 2. Let $U \subset \mathbb{C} \cong \mathbb{R}^2$ be non-empty open and let $\gamma\colon t \mapsto z(t) = x(t) + iy(t)$ be a differentiable path in U. Show that the real tangent

$$\gamma'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$$

can be identified with the holomorphic tangent $z'(t)\frac{\partial}{\partial z}$. Here $\frac{\partial}{\partial z}$ denotes the dual vector of $\mathrm{d}z$.

Exercise 3. Let U be a non-empty open subset of $\mathbb{C}$, let $f\colon U \to \mathbb{C}$ be a holomorphic function and write $\omega = f\mathrm{d}z$.
(i) Prove that ω is *closed*, that is, $\mathrm{d}\omega = 0$.
(ii) Assume that U is contractible. Verify that ω is *exact*, that is, we have $\omega = \mathrm{d}g$ for some holomorphic function $g\colon U \to \mathbb{C}$.

Exercise 4. Give a free abelian subgroup of rank 2 in $\mathbb{C}$ which is not discrete.

Exercise 5. Let $U \subset \mathbb{C}^n$ be a connected open subset. Prove that the set of meromorphic functions $\mathcal{M}(U)$ on U is naturally a field.