Exercise sheet II - 'Jacobians and theta functions' - Fall 2010

Exercise 1. Let $\varphi\colon U \to V$ be a holomorphic map from an open subset of $\mathbb{C}^n$ to an open subset of $\mathbb{C}^n$.
(i) Show that the set

$$R = \{x \in U \,|\, \mathrm{d}\varphi \text{ is not an isomorphism}\}$$

is the zero locus of a holomorphic function. We call R the ramification divisor of φ, if it is different from U.
(ii) Assume that φ is a covering space. Show that R is empty.

Exercise 2. Let τ be an element of $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$. Consider the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ in $\mathbb{C}$ and put $X = \mathbb{C}/\Lambda$. Prove: $\mathrm{End}(X) \supsetneq \mathbb{Z} \Leftrightarrow \tau$ is an element of an imaginary quadratic number field.

Exercise 3. Let $\Lambda = \mathbb{Z} + i\mathbb{Z}$ and put $X = \mathbb{C}/\Lambda$. Show that multiplication by $3 - 4i$ is an isogeny of X to itself. Compute its degree.

In the following V denotes a finite dimensional $\mathbb{C}$-vector space.

Exercise 4. Let H be a hermitian form on V and let E be the associated alternating form. Define $\mathrm{Ker}\, H = \{z \in V \,|\, H(z, w) = 0 \ \forall w \in V\}$ and analogously $\mathrm{Ker}\, E$. Prove that $\mathrm{Ker}\, H = \mathrm{Ker}\, E$.

Exercise 5. Let E be an $\mathbb{R}$-bilinear alternating form on V such that $E(iz, iw) = E(z, w)$ for all $z, w \in V$. Prove that $E(iz, w) + iE(z, w)$ is a hermitian form on V.

Exercise 6. Let H be a hermitian form on V, and put $S = \mathrm{Re}\, H$.
(i) Prove that S is $\mathbb{R}$-bilinear symmetric, and that $S(iz, iw) = S(z, w)$ for all $z, w \in V$.
(ii) Let $S\colon V \times V \to \mathbb{R}$ be $\mathbb{R}$-bilinear symmetric with $S(iz, iw) = S(z, w)$ for all $z, w \in V$. Associate to S a hermitian form H such that $S = \mathrm{Re}\, H$. Prove that S is positive definite iff H is positive definite.

Exercise 7. Let $X = V/\Lambda$ be a complex torus of dimension g. If λ is a basis of Λ and e is a basis of V, the g-by-$2g$ matrix obtained by listing the elements of λ in columns, written on the basis e, is called a period matrix of X.
(i) Show there exist bases of V and Λ with respect to which the period matrix is of the form $(\mathbf{1}|Z)$ with $Z \in M_g(\mathbb{C})$ and $\det \mathrm{Im}\, Z \neq 0$.
(ii) Show that

$$\det \begin{pmatrix} Z & \mathbf{1} \\ \overline{Z} & \mathbf{1} \end{pmatrix} = \det(2i\, \mathrm{Im}\, Z)\,.$$

(iii) Show that conversely the columns of a matrix as in (i) span a lattice in $\mathbb{C}^g$. This gives some justification of the statement that 'the moduli space of complex tori of dimension g is a space of dimension g^2'.