Exercise sheet III - 'Jacobians and theta functions' - Fall 2010

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V, and $X = V/\Lambda$.

Exercise 1. Let $\mathcal{H}$ denote the additive group of Riemann forms on (V, Λ). Write $n = \dim V$. Show that $\mathcal{H}$ is a finitely generated free abelian group of rank $\leq n(2n-1)$.

Exercise 2. Assume $\dim X = 1$. Prove that the additive group of Riemann forms on (V, Λ) is infinite cyclic, and exhibit a generator.

Exercise 3. Let X, Y be isogenous complex tori.
(i) Show that if one of X, Y is an abelian variety, then so is the other.
(ii) Show that a quotient of an abelian variety is an abelian variety.

Exercise 4. Construct a lattice $\Lambda \subset V = \mathbb{C}^2$ such that (V, Λ) has no non-zero Riemann forms.

Exercise 5. Let $\tau \in M_n(\mathbb{C})$ be such that ${}^t\tau = \tau$ and $\operatorname{Im} \tau$ is positive definite. Let $\Lambda = \mathbb{Z}^n + \tau\mathbb{Z}^n \subset V = \mathbb{C}^n$. Prove that $H(z, w) = {}^t z (\operatorname{Im} \tau)^{-1} \overline{w}$ is a positive definite Riemann form on (V, Λ).

Exercise 6. Prove that a complex torus has only countably many subtori.

Exercise 7. Let X, Y be complex tori. Show that there is a natural structure of complex torus on the product $X \times Y$. Show that if X, Y are abelian varieties, then so is $X \times Y$. Express $T_0(X \times Y)$ and $\pi_1(X \times Y)$ in terms of those of X and Y.

Exercise 8. Let Λ_1, Λ_2 be lattices in $\mathbb{C}$, and put $E_1 = \mathbb{C}/\Lambda_1$, $E_2 = \mathbb{C}/\Lambda_2$. Let C_2 be a cyclic group of order 2. Choose $x \in E_1$ such that $\langle x \rangle \cong C_2$, and $y \in E_2$ such that $\langle y \rangle \cong C_2$. Put $K = \langle (x, y) \rangle$ in $E_1 \times E_2$ and define $X = (E_1 \times E_2)/K$.
(i) Show that X is an abelian variety.
(ii) Show that there are natural embeddings of E_1, E_2 in X as abelian subvarieties.
(iii) Show that the images of E_1, E_2 in X intersect non-trivially in X.

Exercise 9. Let K be a totally imaginary quadratic extension of a totally real number field K^+. Let Φ be a type of K. Let O_K be the integral closure of $\mathbb{Z}$ in K. Show that O_K contains an element ξ such that ξ^2 is totally negative and $\operatorname{Im} \phi(\xi) > 0$ for all $\phi \in \Phi$.