Exercise sheet IV - 'Jacobians and theta functions' - Fall 2010

Exercise 1. Let X be a complex manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ an open cover of X. A *refinement* of $\mathcal{U}$ is an open cover $\mathcal{V} = \{V_\beta\}_{\beta\in J}$ together with a map $p\colon J \to I$ such that $V_j \subset U_{p(j)}$ for all $j \in J$.
(i) Let $\mathcal{D} = \{(U_\alpha, f_\alpha)\}$ be a description on $\mathcal{U}$. Let $(\mathcal{V}, p)$ be a refinement of $\mathcal{U}$. Give a definition of the 'restriction' of $\mathcal{D}$ to $(\mathcal{V}, p)$, and prove that this is a description.
(ii) Prove that $\mathcal{D}$ and its restriction to $(\mathcal{V}, p)$ are equivalent.
Note: A complex manifold is *paracompact*, i.e., each open cover has a locally finite refinement. Thus, when $\{(U_\alpha, f_\alpha)\}$ is a description of a Cartier divisor D on X, we may assume that the cover $\{U_\alpha\}$ is locally finite.

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V.

Exercise 2. A holomorphic theta function on (V, Λ) is a holomorphic function $\theta\colon V \to \mathbb{C}$ such that there exist maps $L\colon \Lambda \to V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and $J\colon \Lambda \to \mathbb{C}/\mathbb{Z}$ such that

$$\theta(z + \lambda) = \theta(z)\mathbf{e}(L(\lambda)(z) + J(\lambda))$$

for all $z \in V, \lambda \in \Lambda$.
(i) Assume that θ is non-zero. Prove that L and J are determined by θ.
(ii) Let L, J be associated as above to a non-zero holomorphic theta function θ on (V, Λ). Prove that

$$L(\lambda + \lambda') = L(\lambda) + L(\lambda')$$

and

$$J(\lambda + \lambda') - J(\lambda) - J(\lambda') \equiv L(\lambda, \lambda') \bmod \mathbb{Z}\,.$$

A holomorphic theta function θ is called *trivial* if $\theta(z) = \mathbf{e}(Q(z))$ for some inhomogeneous quadric Q on V.
(iii) Show that a nowhere zero holomorphic theta function is trivial.
A meromorphic theta function on (V, Λ) is the quotient of two holomorphic theta functions for (V, Λ).
(iv) Verify that the set of non-zero meromorphic theta functions on (V, Λ) form a multiplicative group, with the set of trivial theta functions as a subgroup.
The pair (L, J) associated to a nonzero holomorphic theta function θ is called the *type* of θ.
(v) Verify that the set of non-zero theta functions of a fixed type, unioned with $\{0\}$, forms a $\mathbb{C}$-vector space.
Assume the truth of Poincaré's theorem. Let D be an effective Cartier divisor on $X = V/\Lambda$. We put

$$\mathcal{L}(D) = \{f \in \mathcal{M}(X)^* \,|\, \mathrm{div}\,(f) + D \geq 0\} \cup \{0\}\,.$$

This is a $\mathbb{C}$-vector space. Let θ be a meromorphic theta function on (V, Λ) such that $\pi^*D = \mathrm{div}\,\theta$; here $\pi\colon V \to X$ is the canonical projection.
(vi) Show that in fact θ is holomorphic.
Let (L, J) be the type of θ and let $\mathrm{Th}(L, J)$ be the vector space associated to (L, J) by (v).
(vii) Prove that $f \mapsto f \cdot \theta$ defines a $\mathbb{C}$-linear isomorphism $\mathcal{L}(D) \cong \mathrm{Th}(L, J)$.