Exercise sheet IX - 'Jacobians and theta functions' - Fall 2010

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V, and $X = V/\Lambda$.

Exercise 1. Let θ be a meromorphic theta function on (V, Λ) associated to a divisor D on X. Let $x \in V$. Assume that D is invariant under translation by $\overline{x}$, i.e. $D = D + \overline{x}$.
(i) Show that $\theta(z - x)/\theta(z)$ is a trivial theta function.
(ii) Give the automorphy factor of $\theta(z - x)/\theta(z)$ in terms of x and the type of θ.
Assume that the associated Riemann form of D is positive definite. Then it can be shown that $\Lambda + \mathbb{Z} \cdot x$ is a lattice in V.
(iii) Prove that θ is a theta function on $(V, \Lambda + \mathbb{Z} \cdot x)$.

Exercise 2. Let D be an effective divisor on X such that the associated Riemann form is positive definite. Assuming the truth of the Lefschetz embedding theorem, verify that $\mathcal{L}(mD)$ gives rise to a projective embedding for all $m \in \mathbb{Z}_{\geq 3}$.

Assume that $V = \mathbb{C}$. Let σ and $\wp$ be the Weierstrass functions associated to Λ.

Exercise 3. Show the "addition formula"

$$\wp(z) - \wp(w) = -\frac{\sigma(z - w)\sigma(z + w)}{\sigma^2(z)\sigma^2(w)}$$

for all $z, w \in \mathbb{C} \setminus \Lambda$.

Exercise 4. (i) Find the zeroes of $\wp'(z)$ on X. Hint: $\wp'$ has a pole of order 3 at o, and no other poles. Hence, it has 3 zeroes, counted with multiplicity.
(ii) Show the formula

$$\wp'(z) = -\frac{\sigma(2z)}{\sigma^4(z)}$$

for all $z \in \mathbb{C} \setminus \Lambda$.

Exercise 5. Let $\alpha_i, \alpha_j, \alpha_k$ be the non-trivial 2-torsion points on X.
(i) Show that $\wp(\alpha_i), \wp(\alpha_j), \wp(\alpha_k)$ are the distinct roots of the polynomial $4x^3 - g_2x - g_3$. Here $g_2, g_3 \in \mathbb{C}$ are associated to X as in class.
(ii) Show that the formula

$$\wp(\alpha_i) - \wp(\alpha_j) = -\frac{\sigma(\alpha_k)^2}{\sigma(\alpha_i)^2\sigma(\alpha_j)^2}$$

holds.