Exercise sheet V - 'Jacobians and theta functions' - Fall 2010

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V, and $X = V/\Lambda$.

Exercise 1. Prove *Cousin's Theorem*: let D be a Cartier divisor on V. Then D is principal. Hint: imitate the first half of the proof of Poincaré's Theorem.

Exercise 2. An *automorphy factor* on (V, Λ) is a holomorphic function $j\colon V \times \Lambda \to \mathbb{C}^*$ such that

$$j(z, \lambda + \lambda') = j(z + \lambda, \lambda') \cdot j(z, \lambda)$$

for all $\lambda, \lambda' \in \Lambda$ and all $z \in V$.
(i) Show that set J of automorphy factors on (V, Λ) is a multiplicative group.
An automorphy factor j is called a *boundary* if there is a holomorphic function $g\colon V \to \mathbb{C}^*$ such that $j(z, \lambda) = g(z + \lambda)/g(z)$ for all $\lambda \in \Lambda$, all $z \in V$.
(ii) Show that the boundaries form a subgroup of J.
We call 2 automorphy factors cohomologous if they (multiplicatively) differ by a boundary. Let D be an effective divisor on X and let $f\colon V \to \mathbb{C}$ be holomorphic such that $\pi^* D = \operatorname{div} f$, where $\pi\colon V \to X$ is the projection.
(iii) Show there exists an automorphy factor j_f such that

$$f(z + \lambda) = f(z) \cdot j_f(z, \lambda)$$

for all $\lambda \in \Lambda$, all $z \in V$.
(iv) Verify that Poincaré's Theorem is equivalent to the statement that j_f is cohomologous to an automorphy factor of the form

$$(z, \lambda) \mapsto \mathbf{e}(L(\lambda)(z) + J(\lambda))$$

for some $L\colon \Lambda \to V^*$ and some $J\colon \Lambda \to \mathbb{C}/\mathbb{Z}$.

Exercise 3. Let θ be a *holo*morphic theta function whose associated Riemann form is zero. Prove that θ is a trivial theta function.

Exercise 4. Let $\alpha\colon \Lambda \to U(1)$ be the quasi-character associated to a normalized theta function on (V, Λ). Show that $\alpha(0) = 1$ and $\alpha(-\lambda) = \alpha(\lambda)^{-1}$ for all $\lambda \in \Lambda$.

Exercise 5. *(Riemann's theta function)* Assume $V = \mathbb{C}^n$ and $\Lambda = \mathbb{Z}^n + \tau\mathbb{Z}^n$ where τ is a symmetric n-by-n matrix such that $\operatorname{Im}\tau$ is positive definite. We define $H = H(z, w) = {}^t z(\operatorname{Im}\tau)^{-1} w$; this is a (positive definite) Riemann form on (V, Λ). For $z \in V$ we put

$$\theta(z) = \theta(z, \tau) = \sum_{m \in \mathbb{Z}^n} \exp(\pi i {}^t m \tau m + 2\pi i {}^t m z)\,.$$

(i) Show that θ is a holomorphic function on V.
(ii) Verify that θ satisfies the functional equation

$$\theta(z + l + \tau m) = \theta(z) \exp(-2\pi i {}^t m z - \pi i {}^t m \tau m)$$

for all $z \in V$, all $l, m \in \mathbb{Z}^n$. Hence, θ is a holomorphic theta function on (V, Λ).
(iii) Verify that the associated Riemann form is H.
(iv) Show that the normalized theta function associated to θ is

$$\theta_1(z) = \theta(z) \exp(\frac{\pi}{2} {}^t z (\operatorname{Im} \tau)^{-1} z) .$$

Show that the corresponding "K-function" is $K(l + \tau m) = {}^t l m / 2$ for $l, m \in \mathbb{Z}^n$.