Exercise sheet VI - 'Jacobians and theta functions' - Fall 2010

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V, and $X = V/\Lambda$.

Exercise 1. Let D be an effective divisor on X, let θ be a normalized theta function associated to D and let H and α be the Riemann form resp. the quasi-character associated to θ. Prove that α is identically equal to 1 on $\Lambda \cap \operatorname{Ker} H$.

Exercise 2. Let D be an effective divisor on X, and let H be the Riemann form associated to D. Let D' be the induced divisor on $X' = V'/\Lambda'$ where $V' = V/\operatorname{Ker} H$ and $\Lambda' = \Lambda/(\Lambda \cap \operatorname{Ker} H) \subset V'$. Let $p\colon X \to X'$ be the projection. Verify that $p^*\colon \mathcal{L}(D') \to \mathcal{L}(D)$ is an isomorphism (recall the notation $\mathcal{L}$ from Exercise IV.2).
Hint: use Exercise IV.2.(vii).

Exercise 3. Prove that the assignments

$$\operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}) \rightleftarrows \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$$

given by

$$k \mapsto \operatorname{Im} k$$

(from left to right) and

$$f \mapsto (z \mapsto f(iz) + if(z))$$

(from right to left) are $\mathbb{R}$-linear isomorphisms, each other's inverse.

Exercise 4. Assume $\dim_{\mathbb{C}} V = n$. Let E be a $\mathbb{Z}$-valued alternating bilinear form on Λ. Show that there exist precisely 2^{2n} maps $\alpha\colon \Lambda \to \{\pm 1\}$ such that

$$\alpha(\lambda + \lambda') = \alpha(\lambda)\alpha(\lambda')(-1)^{E(\lambda,\lambda')}$$

for all $\lambda, \lambda' \in \Lambda$.