Exercise sheet VII - 'Jacobians and theta functions' - Fall 2010

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V, and $X = V/\Lambda$.

Exercise 1. Let E be a $\mathbb{Z}$-valued alternating form on $\Lambda \times \Lambda$.
(i) Let $\Lambda_1 \subset \Lambda$ be a lattice. Show that $\mathrm{Pf}(E|_{\Lambda_1}) = [\Lambda \colon \Lambda_1]\mathrm{Pf}(E)$.
Assume that E is non-degenerate.
(ii) Prove that there are only finitely many lattices $\Lambda_2 \supset \Lambda$ contained in V such that E extends to a $\mathbb{Z}$-valued form on $\Lambda_2 \times \Lambda_2$.

Exercise 2. Let H be a positive semi-definite hermitian form on V. Prove that the equality $\mathrm{Ker}\, H = \{z \in V \mid H(z, z) = 0\}$ holds.

Exercise 3. *(Abelianization of a complex torus)* Let $K \subset V$ be the intersection of all $\mathrm{Ker}\, H$, where H runs through the set $\mathcal{H}$ of positive semi-definite Riemann forms on (V, Λ).
(i) Show that $K = \mathrm{Ker}\, H_1 \cap \ldots \cap \mathrm{Ker}\, H_r$ for a finite subset $\{H_1, \ldots, H_r\}$ of $\mathcal{H}$.
Let $H = H_1 + \cdots + H_r$.
(ii) Prove that H is positive semi-definite, and that $K = \mathrm{Ker}\, H$.
Let $V' = V/\mathrm{Ker}\, H$ and let Λ' be the image of Λ in V'. We know that $X' = V'/\Lambda'$ is an abelian variety and that H induces a positive definite Riemann form on (V', Λ'). Let $p \colon X \to X'$ be the projection.
(iii) Show that the following universal property holds: every morphism of tori $X \to Y$ with Y an abelian variety factors uniquely through p.
We call X' the *abelianization* of X.
(iv) Show that p^* induces bijections $\mathrm{Div}\, X' \xrightarrow{\sim} \mathrm{Div}\, X$ and $\mathcal{M}(X') \xrightarrow{\sim} \mathcal{M}(X)$.
In particular, if X admits no non-zero Riemann forms then the abelianization of X is a point, and all meromorphic functions on X are constants.

Exercise 4. Let (H, α) be an "Appell-Humbert datum" on (V, Λ), that is:
H is a Riemann form on (V, Λ);
$\alpha \colon \Lambda \to U(1)$ is a map satisfying the equality

$$\alpha(\lambda + \lambda') = \alpha(\lambda)\alpha(\lambda')(-1)^{E(\lambda, \lambda')}$$

for all λ, λ' in Λ, where $E = \mathrm{Im}\, H$.
Show that the following two assertions are equivalent:
(a) there exists a holomorphic normalized theta function on (V, Λ) with associated Riemann form H and associated quasi-character α;
(b) H is positive semi-definite, and $\alpha \equiv 1$ on $\Lambda \cap \mathrm{Ker}\, H$.

Exercise 5. Let D be an effective divisor on X with associated Riemann form H. Show the equality

$$\dim \mathcal{L}(X, r \cdot D) = r^{\dim V - \dim \mathrm{Ker}\, H} \dim \mathcal{L}(X, D)$$

for all $r \in \mathbb{Z}_{>0}$.

Exercise 6. Let $f\colon X \to Y$ be an isogeny of abelian varieties. Let D be an effective divisor on Y whose associated Riemann form is positive definite. Show that

$$\dim \mathcal{L}(X, f^*D) = (\deg f) \dim \mathcal{L}(Y, D)\,.$$

Hint: use Exercise 1 of this sheet.

Exercise 7. Find the elementary divisors and a symplectic basis for the alternating form E on $\mathbb{Z}^4$ given by the matrix:

$$\begin{pmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & -5 & -4 \\ 3 & 5 & 0 & -2 \\ 0 & 4 & 2 & 0 \end{pmatrix}.$$