Exercise sheet VIII - 'Jacobians and theta functions' - Fall 2010

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V, and $X = V/\Lambda$. Let θ be a meromorphic theta function on (V, Λ) associated to a divisor D on X.

Exercise 1. *(Theorem of the square)* (i) Show that for any $x, y \in V$, the meromorphic function

$$\frac{\theta(z-x)\theta(z-y)}{\theta(z-x-y)\theta(z)}$$

is Λ-periodic. Hint: compute its automorphy factor.
(ii) Conclude that for any $x, y \in X$ the divisors

$$D_x + D_y \quad \text{and} \quad D_{x+y} + D$$

are linearly equivalent. Here D_x is a shorthand of the translate $D + x$ of D by x.

Exercise 2. *(Theorem of the cube)* Show that for any $x, y \in V$ the meromorphic function

$$\frac{\theta(z-x)\theta(z-y)\theta(z+x+y)}{\theta(z)^3}$$

is Λ-periodic.

Exercise 3. Let $x \in V$. Let (L, J) be the type of θ.
(i) Show that $\theta(z-x)$ is a meromorphic theta function on (V, Λ).
(ii) Compute the automorphy factor j_x for the quotient $\theta(z-x)/\theta(z)$.
(iii) Verify that j_x can be viewed as a homomorphism $\Lambda \to \mathbb{C}^*$, hence as the exp of a homomorphism $\Lambda \to \mathbb{C}$.
(iv) Let x vary through V. Verify that the natural map $V \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$ thus obtained is $\mathbb{C}$-linear.
Assume that θ is normalized, and let H be the associated Riemann form.
(v) Verify that $j(z, \lambda) = \mathbf{e}(-\frac{1}{2i}H(z, \lambda))$.

Exercise 4. Let f be a meromorphic function on V such that for some $z \in V$

$$f(z-u) + f(z-v) + f(z+u+v) = 0$$

for all $u, v \in V$. Show that $u \mapsto f(z-u)$ is a $\mathbb{C}$-linear map on V.

Exercise 5. Show that the morphism $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ given by $(t : u) \mapsto (t^2u : t^3 : u^3)$ separates points but does not separate tangent vectors at $(0 : 1)$.