**Exercise sheet XI - 'Jacobians and theta functions' - Fall 2010**

In the following, $V$ denotes a finite dimensional $\mathbb{C}$-vector space, $\Lambda$ denotes a lattice in $V$, and $X = V/\Lambda$.

**Exercise 1.** Prove that the assignments

$$\overline{\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})} \rightleftarrows \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{R})$$

given by

$$k \mapsto \operatorname{Im} k$$

(from left to right) and

$$f \mapsto (z \mapsto -f(iz) + if(z))$$

(from right to left) are $\mathbb{R}$-linear isomorphisms, each other's inverse.

We put

$$\widehat{V} = \overline{\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})}\,, \quad \widehat{\Lambda} = \{f \in \widehat{V} \mid (\operatorname{Im} f)(\Lambda) \subset \mathbb{Z}\}\,, \quad \widehat{X} = \widehat{V}/\widehat{\Lambda}\,.$$

It was shown in class that $\widehat{X}$ has a natural structure of complex torus, the *dual complex torus* of $X$.

**Exercise 2.** (i) Show that the assignment

$$z \mapsto \big(f \mapsto \overline{f}(z)\big)$$

is a $\mathbb{C}$-linear isomorphism $V \xrightarrow{\sim} \widehat{\widehat{V}}$.
(ii) Show that the map from (i) induces an isomorphism of complex tori can: $X \xrightarrow{\sim} \widehat{\widehat{X}}$.

**Exercise 3.** Let $f\colon X \to Y$ be a morphism of complex tori.
(i) Exhibit a natural morphism $\widehat{f}\colon \widehat{Y} \to \widehat{X}$ of dual complex tori.
(ii) Show that $f$ is an isogeny $\Leftrightarrow$ $\widehat{f}$ is an isogeny.
(iii) Show that if $f$ is an isogeny, then $\deg f = \deg \widehat{f}$.

**Exercise 4.** Let $f\colon X \to Y$ be a morphism of complex abelian varieties.
(i) Show that pullback $f^*\colon \operatorname{Div} Y \to \operatorname{Div} X$ induces a natural map $f^*\colon \operatorname{Pic}^0 Y \to \operatorname{Pic}^0 X$.
(ii) Show that under the natural identification $\operatorname{Pic}^0 X \cong \widehat{X}$ discussed in class, the map $f^*$ coincides with the map $\widehat{f}$ from Exercise 3.

Recall that for any divisor $D$ on a complex abelian variety $X$ we have a group homomorphism $\varphi_D\colon X \to \operatorname{Pic}^0 X$ given by $x \mapsto$ the class of $(D + x) - D$.

**Exercise 5.** Let $f\colon X \to Y$ be a morphism of complex abelian varieties. Let $D$ be a divisor on $Y$. Show that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ \downarrow{\scriptstyle \varphi_{f^*D}} & & \downarrow{\scriptstyle \varphi_D} \\ \operatorname{Pic}^0 X & \xleftarrow{\;f^*\;} & \operatorname{Pic}^0 Y \end{array}$$

commutes.

**Exercise 6.** A morphism $f\colon X \to \widehat{X}$ is called *symmetric* if $\widehat{f} \circ \mathrm{can}\colon X \to \widehat{X}$ is the same as $f$. Here $\mathrm{can}\colon X \to \widehat{\widehat{X}}$ is the canonical isomorphism from Ex. 2. Show that for any divisor $D$ on a complex abelian variety $X$ the morphism $\varphi_D$ is symmetric.