Exercise sheet XII - 'Jacobians and theta functions' - Fall 2010

In the following, V denotes a finite dimensional $\mathbb{C}$-vector space, Λ denotes a lattice in V, and $X = V/\Lambda$.

Exercise 1. (Weil pairing) Let $D = D_0$ be an effective divisor on X giving rise to a principal polarization, i.e. the associated Riemann form H is positive definite and $E = \operatorname{Im} H$ has $\mathrm{Pf}_\Lambda(E) = 1$. Let $n \in \mathbb{Z}_{>0}$. Write $X[n] = \{x \in X \mid n \cdot x = 0\}$, the group of n-torsion points of X. Let $x \in X[n]$.

(i) Prove that $n \cdot D_x$ and $n \cdot D_0$ are linearly equivalent divisors. Here we write D_x as a shorthand for the translated divisor $D + x$.

(ii) Show that $K(n \cdot D) = X[n]$.

Let $\mathcal{G}_{D,n}$ be the set of pairs (f, x) where $x \in X[n]$ and $f \in \mathcal{M}(X)^*$ is such that $\operatorname{div}(f) = n \cdot D_x - n \cdot D_0$.

(iii) Prove that $\mathcal{G}_{D,n}$ becomes a group if we put

$$(f, x) \cdot (g, y) = (f \cdot t_{-x}^* g, x + y)\,, \quad (f, x)^{-1} = (t_x^* f^{-1}, -x)\,.$$

Here $(t_x^* f)(y) = f(x + y)$. We call $\mathcal{G}_{D,n}$ the *theta group* associated to D, n.

(iv) Show that there is a natural exact sequence of groups

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{G}_{D,n} \longrightarrow K(n \cdot D) \longrightarrow 0\,.$$

such that $\mathbb{C}^*$ becomes the center of $\mathcal{G}_{D,n}$.

(v) Show that the commutator pairing $[,] \colon \mathcal{G}_{D,n} \times \mathcal{G}_{D,n} \to \mathcal{G}_{D,n}$ induces an alternating form $e_n \colon X[n] \times X[n] \to \mu_n$. Here μ_n is the group of n-th roots of unity in $\mathbb{C}^*$. This pairing on $X[n]$ is called the *Weil pairing* associated to D.

(vi) Let $E \colon \Lambda \times \Lambda \to \mathbb{Z}$ as above be the non-degenerate alternating form associated to D. Let $x', y' \in \frac{1}{n}\Lambda$ be lifts of $x, y \in X[n] = \frac{1}{n}\Lambda/\Lambda$. Show that $e_n(x, y) = \mathbf{e}(nE(x', y'))$.

Hint: let θ be a normalized theta function for D. Let $f \in \mathcal{M}(X)^*$ be such that $(f) = n \cdot D_x - n \cdot D_0$. Then up to a multiplicative constant f is given by ξ^n where ξ is the meromorphic theta on (V, Λ) given by

$$\xi(z) = \mathbf{e}(\frac{1}{2i} H(x', z)) \cdot \theta(z - x')/\theta(z)\,.$$

(vii) Prove that e_n is non-degenerate, i.e. has trivial left and right kernel.