

Additional exercises Topics in Geometry, Fall 2006

Rings are always supposed to be commutative with 1.

Exercise 1. Let k be a field and let n be a positive integer. The *ring of polynomial functions* $O(k^n)$ on k^n is the ring of functions $f : k^n \rightarrow k$ such that there exists a polynomial g in $k[x_1, \dots, x_n]$ with $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ for all $(x_1, \dots, x_n) \in k^n$.

(i) Determine $O(k)$ for k a finite field with q elements.

(ii) Assume that k is an infinite field. Prove that $O(k^n) \cong k[x_1, \dots, x_n]$.

Exercise 2. Let A be a ring and let I be an ideal of A . The *radical* \sqrt{I} of I is defined as

$$\sqrt{I} = \{a \in A : \text{there exists } n > 0 \text{ such that } a^n \in I\}.$$

Prove:

(i) $\sqrt{I} \supseteq I$;

(ii) \sqrt{I} is an ideal of A ;

(iii) $\sqrt{\sqrt{I}} = \sqrt{I}$;

(iv) \sqrt{I} equals the intersection of all prime ideals containing I . (Hint: in order to prove the difficult inclusion, take an arbitrary $a \notin \sqrt{I}$. We want the existence of a prime ideal $\mathfrak{p} \supseteq I$ such that $a \notin \mathfrak{p}$. Consider the collection of ideals J of A that contain I but have $a \notin \sqrt{J}$. Apply Zorn's Lemma to this (clearly non-empty) collection. Any maximal element of it will be a prime ideal.)

Exercise 3. Let X be an affine variety. For f in $A(X)$ define $D(f) = \{p \in X : f(p) \neq 0\}$ and $Z(f) = X \setminus D(f) = \{p \in X : f(p) = 0\}$. Prove:

(i) $X = \cup_{i \in I} D(f_i) \Leftrightarrow \cap_{i \in I} Z(f_i) = \emptyset \Leftrightarrow$ the f_i generate $A(X)$ as an ideal.

(ii) The $D(f)$ with f running through $A(X)$ form a basis for the Zariski topology on X .

(iii) X is quasi-compact, i.e., every covering of X with open subsets has a finite subcovering.

Exercise 4. Suppose I is an ideal of a ring R . Show that if \sqrt{I} is finitely generated, then for some integer N we have $\sqrt{I}^N \subseteq I$. Conclude that in a noetherian ring, the ideals I and J have the same radical iff there is some integer N such that $I^N \subseteq J$ and $J^N \subseteq I$. Use Hilbert's Nullstellensatz to deduce that if $I, J \subseteq A = k[x_1, \dots, x_n]$ are ideals and k is an algebraically closed field, then $Z(I) = Z(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some integer N .

Exercise 5. (Integrality) Let A be a subring of a ring B . We call an element $b \in B$ *integral* over A if there exist $a_0, \dots, a_{m-1} \in A$ such that $b^m + a_{m-1}b^{m-1} + \dots + a_0 = 0$. We call B integral over A if every element of B is integral over A .

(i) Let $b \in B$. Prove that the following statements are equivalent: (1) b is integral over A ; (2) $A[b]$ is a finitely generated A -module; (3) $A[b]$ is contained in a subring C of B such that C is a finitely generated A -module. (Hint: in order to prove (3) \Rightarrow (1), write $C = Af_1 + \dots + Af_n$. For each i there are $c_{ij} \in A$ such that $b \cdot f_i = \sum_j c_{ij} \cdot f_j$. In other words, the vector ${}^t(f_1, \dots, f_n)$ is in the kernel of the matrix $m_{ij} := b \cdot \delta_{ij} - c_{ij}$. Use the adjoint matrix of m_{ij} to get from this that $d \cdot f_i = 0$ for all i , where d is the determinant of m_{ij} . Since 1 is a linear combination of the f_i , we get then that $d = 0$. Finish by observing that d is a monic polynomial in b with coefficients in A .)

(ii) Let b_1, \dots, b_n be elements of B , each integral over A . Prove that the ring $A[b_1, \dots, b_n]$ is

a finitely generated A -module.

(iii) Prove that the set of elements of B which are integral over A is a subring of B containing A .

(iv) Let $A \subseteq B \subseteq C$ be rings, and assume B integral over A , and C integral over B . Prove that C is integral over A .

(v) Let B be a domain and let A be a subring of B such that B is integral over A . Prove: B is a field $\Leftrightarrow A$ is a field.

Exercise 6. (Localisation) Let A be a ring and let $S \subseteq A$ be a multiplicative subset, i.e., a subset S of A with $1 \in S$ and closed under multiplication. Define a relation \equiv on $A \times S$ as follows: let $(a, s) \equiv (b, t) \Leftrightarrow u \cdot (at - bs) = 0$ for some $u \in S$.

(i) Prove that \equiv is an equivalence relation.

We denote by $S^{-1}A$ the set of equivalence classes for \equiv .

(ii) Prove that $S^{-1}A$ has a natural ring structure, and that there is a natural ring homomorphism $\phi : A \rightarrow S^{-1}A$.

For $f \in A$ one usually denotes $S^{-1}A$ for $S = \{1\} \cup \{f^n\}_{n \in \mathbb{Z}_{>0}}$ by A_f . For example, if $A = \mathbb{Z}$ and $f = 2$ then A_f is the ring of rational numbers a/b with $a, b \in \mathbb{Z}$ such that b is a power of 2.

(iii) Prove that A_f is the zero ring $\Leftrightarrow f$ is nilpotent. Hence the map ϕ of (ii) need not be injective.

For \mathfrak{p} a prime ideal of A one usually denotes $S^{-1}A$ for $S = A - \mathfrak{p}$ by $A_{\mathfrak{p}}$. For example, if $A = \mathbb{Z}$ and $\mathfrak{p} = (3)$ then $A_{\mathfrak{p}}$ is the ring of rational numbers a/b with $a, b \in \mathbb{Z}$ such that b is not divisible by 3.

(iv) Prove that $A_{\mathfrak{p}}$ has a unique maximal ideal.

A ring with a unique maximal ideal is called a *local ring*.

(v) Prove that $S^{-1}A$ and ϕ satisfy the following universal property: let $g : A \rightarrow B$ be a ring homomorphism such that $g(s)$ is a unit in B for all s in S . Then there is a unique ring homomorphism $h : S^{-1}A \rightarrow B$ such that $g = h \cdot \phi$.

(vi) Prove that if A is a subring of a ring B , and B is integral over A , then $S^{-1}B$ is integral over $S^{-1}A$.

Exercise 7. If A is a local ring with maximal ideal \mathfrak{m} , we call the field A/\mathfrak{m} the *residue field* of A .

(i) If B is a ring and \mathfrak{p} is a prime ideal of B , prove that the residue field of $B_{\mathfrak{p}}$ is isomorphic to the field of fractions of the domain B/\mathfrak{p} .

(ii) Prove that a ring A is local iff the set of non-units of A is an ideal of A .

(iii) Prove that a local ring is not a direct sum of two rings.

Exercise 8. Let X be an affine variety. For non-constant f in $A(X)$ let $Y = D(f)$ be the corresponding quasi-affine variety. Prove that Y is (naturally isomorphic to) an affine variety, and describe the coordinate ring of Y . (Use localisation. See also Lemma I.4.2 in [HAG].)

Exercise 9. Let k be an algebraically closed field and let $f_1, \dots, f_r \in k[x_1, \dots, x_n]$. Let g in $k[x_1, \dots, x_n]$ be such that $f_1(x_1, \dots, x_n) = \dots = f_r(x_1, \dots, x_n) = 0 \Rightarrow g(x_1, \dots, x_n) = 0$. Rephrase our proof that $g \in \sqrt{(f_1, \dots, f_r)}$ (the Hilbert Nullstellensatz!) as follows.

(i) Let $A = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ and consider the localisation map $A \rightarrow A_{\bar{g}}$. Use the

Weak Nullstellensatz to prove that $A_{\bar{g}}$ is the zero ring.

(ii) Conclude that \bar{g} is nilpotent in A .

(iii) Conclude that $g \in \sqrt{(f_1, \dots, f_r)}$.

Exercise 10. (The fibres of a morphism) Let X, Y be affine varieties and let $f : X \rightarrow Y$ be a dominant morphism. Then f induces an injective homomorphism $f^* : A(Y) \rightarrow A(X)$. Let $p \in Y$ be a point, with corresponding maximal ideal \mathfrak{m}_p in $A(Y)$. Prove that the points in $f^{-1}(p)$ naturally correspond to the maximal ideals of $A(X)$ that contain $\mathfrak{m}_p A(X)$. (Remark: one gets more refined information if one considers the ring $A(X)/\mathfrak{m}_p A(X)$ instead of the set of its maximal ideals. This is one of the ideas that motivate the theory of *schemes*, cf. [HAG], Chapter II, especially pp. 89–90.)

Exercise 11. Draw a picture of the solution set of the equation $y^2 = (x - 1)x(x + 1)$ in \mathbb{R}^2 . Show that $Z(y^2 - (x - 1)x(x + 1))$ in \mathbb{C}^2 is irreducible for the Zariski topology.

Exercise 12. Determine the irreducible components of $Z(y^4 - x^6, y^3 - xy^2 - yx^3 + x^4)$ in \mathbb{A}^2 .

Exercise 13. Let X be a Hausdorff space. Determine the irreducible subspaces of X .

Exercise 14. Prove that $X = Z(x^2 - y, y^2 - z)$ in \mathbb{A}^3 is irreducible, and that $X \cong \mathbb{A}^1$. Prove that $Z(x^2 - y^3) \subseteq \mathbb{A}^2$ is not isomorphic with \mathbb{A}^1 . Give a bijective morphism from \mathbb{A}^1 to $Z(x^2 - y^3)$.

Exercise 15. Let k be an algebraically closed field. Describe the prime ideals of $k[x, y]$. Describe the algebraic subsets of \mathbb{A}^2 .

Exercise 16. Describe the maximal ideals of $k[x]$ for various fields k such as $k = \mathbb{R}$, k a finite field, ...

Exercise 17. Let k be an algebraically closed field of characteristic $p > 0$, and let $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ be polynomials with coefficients in a finite field \mathbb{F}_q . Let $X = Z(f_1, \dots, f_r) \subseteq \mathbb{A}^n$.

(i) Show that the map $F_q : X \rightarrow X$ given by $(x_1, \dots, x_n) \mapsto (x_1^q, \dots, x_n^q)$ is a morphism. We call this map the q -th *Frobenius map*.

(ii) Show that F_q is bijective.

(iii) Show that F_q is not an isomorphism unless X is a point.

(iv) Describe the fixed point set of F_q and prove that it is finite.

Exercise 18. Let S be a graded ring. An ideal $\mathfrak{a} \subseteq S$ is called *homogeneous* if every $f \in \mathfrak{a}$ has all its components in \mathfrak{a} . Prove:

(i) An ideal is homogeneous iff it can be generated by homogeneous elements.

(ii) The sum, product, intersection and radical of homogeneous ideals are homogeneous.

(iii) The homogeneous ideal \mathfrak{a} is prime iff for any two homogeneous elements $f, g \in S$ one has that $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

Exercise 19. (Noetherian modules) Let A be a ring and let M be an A -module.

(i) Prove that the following conditions on M are equivalent: every submodule of M is finitely generated; every ascending chain $M_1 \subseteq M_2 \subseteq \dots$ of submodules of M becomes stationary; every non-empty set of submodules of M has a maximal element. A module satisfying these

conditions is called *noetherian*.

(ii) Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of A -modules. Prove: M is noetherian iff M' and M'' are noetherian.

(iii) If M_i for $i = 1, \dots, n$ are noetherian A -modules, then so is $\bigoplus_{i=1}^n M_i$. Prove this.

(iv) Assume that A is a noetherian ring and that M is a finitely generated A -module. Prove that M is noetherian.

Exercise 20. Let k be a field, let n be a positive integer and let \mathbb{P}^n be projective n -space over k . Prove that there is a natural decomposition

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \dots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0$$

in disjoint subsets. Compute the number of elements of \mathbb{P}^n for k a finite field of q elements.

Exercise 21. For any $d \in \mathbb{Z}_{\geq 0}$, let $S_d \subseteq S = k[x_0, \dots, x_n]$ be the k -vector space of homogeneous polynomials of degree d . Prove that $\dim_k S_d = \binom{d+n}{n}$.

Exercise 22. Let k be an algebraically closed field. Prove that $PGL(2, k)$ acts via automorphisms on \mathbb{P}^1 , and that the action is 3-transitive. Prove also that if a projective transformation of \mathbb{P}^1 has three fix points, then it is the identity.

Exercise 23. More generally, prove that $PGL(n+1, k)$ acts via automorphisms on \mathbb{P}^n .

We call a collection of $k \geq n+2$ distinct points in \mathbb{P}^n to be *in general position* if no $n+1$ among them lie in a hyperplane. Let p_0, \dots, p_{n+1} be $n+2$ points in general position in \mathbb{P}^n . Prove that we can choose homogeneous coordinates such that $p_0 = (1 : 0 : \dots : 0)$, $p_1 = (0 : 1 : 0 : \dots : 0)$, \dots , $p_n = (0 : \dots : 0 : 1)$ and $p_{n+1} = (1 : 1 : \dots : 1)$.

Choose five points in general position in \mathbb{P}^2 . Show that there is a unique non-degenerate conic passing through them.

Exercise 24. (Classification of conics over \mathbb{C}) Recall that a conic in the affine real plane \mathbb{R}^2 (that is, the locus in \mathbb{R}^2 defined by a quadratic equation with real coefficients) belongs to one of the following eight types:

- (a) the empty set (as with $x^2 + y^2 + 1 = 0$);
- (b) a single point (as with $x^2 + y^2 = 0$);
- (c) a 'double line' ($x^2 = 0$);
- (d) the union of two incident lines ($xy = 0$);
- (e) the union of two parallel lines ($x(x-1) = 0$);
- (f) a parabola ($y - x^2 = 0$);
- (g) a hyperbola ($xy - 1 = 0$);
- (h) an ellipse ($x^2 + 2y^2 - 1 = 0$).

Any two examples of one of these types differ only by a (real) projective linear transformation.

(i) Show that in the affine complex plane $\mathbb{A}_{\mathbb{C}}^2$ there are only five types of conics: types (a) and (b) disappear, and types (g) and (h) coincide.

(ii) Show that in the projective complex plane $\mathbb{P}_{\mathbb{C}}^2$ there are only three types of conics: they are represented by types (c), (d) and (h) from the above list. (Hint: this is a classification by the *rank* of a conic, where the rank of a quadratic form $\sum_i a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_ix_j$ is defined by the rank of the symmetric matrix (a_{ij}) .)

(iii) Show that the different types in (i) correspond to the relative position of the conic and the line at infinity. More precisely, a parabola is a rank-3 conic tangent to the line at infinity, while an ellipse/hyperbola is a rank-3 conic meeting the line at infinity in two distinct points.
 (iv) For a real conic C , complex conjugation acts in a natural way on the set of its complex-valued points. Assume that C belongs to type (g) or (h). Prove: C is a hyperbola iff its points at infinity are fixed under complex conjugation (i.e., real), and: C is an ellipse iff its points at infinity are conjugate.

Exercise 25. Let $A = (a_0 : a_1 : a_2)$ and $B = (b_0 : b_1 : b_2)$ be distinct points in \mathbb{P}^2 . Give an equation for the line AB passing through A and B .

Exercise 26. (Pascal's Theorem) Let C be a rank-3 conic in \mathbb{P}^2 , and let A, B, C, A', B', C' be six distinct points on C . Prove that the points $P = AB' \cap A'B$, $Q = AC' \cap A'C$ and $R = BC' \cap B'C$ are collinear.

Exercise 27. Give an isomorphism between \mathbb{P}^1 and $Z(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$. Parametrise all integer solutions to the equation $x^2 + y^2 = z^2$.

Exercise 28. (Closedness can be checked locally) Let X be a topological space and $Y \subseteq X$. Prove: Y is closed in $X \Leftrightarrow$ there is a covering $X = \cup_i U_i$ with open subsets such that $Y \cap U_i$ is closed in U_i for all $i \Leftrightarrow$ for any covering $X = \cup_i U_i$ with open subsets one has $Y \cap U_i$ closed in U_i for all i .

Exercise 29. Let Y be a variety over an algebraically closed field k , let $U \subseteq Y$ be a non-empty open subset and $p \in U$ a point. Prove that the inclusion $U \hookrightarrow Y$ gives rise to natural isomorphisms $O_{p,U} \cong O_{p,Y}$ and $K(U) \cong K(Y)$ of k -algebras.

Exercise 30. Consider the "folium of Descartes" $X = Z(xyz - x^3 - y^3) \subseteq \mathbb{P}^2$. Prove that X is birationally equivalent with \mathbb{P}^1 . Are they isomorphic?

Exercise 31. Consider the quadric $X = Z(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$. Prove that X is birationally equivalent with \mathbb{A}^2 . Are they isomorphic?

Exercise 32. Prove that \mathbb{P}^2 is birationally equivalent with $\mathbb{P}^1 \times \mathbb{P}^1$. Are they isomorphic?

Exercise 33. (i) Let X be a variety that is both projective and affine. Prove that X consists of a single point.

(ii) Let X be a projective variety, Y an affine variety, and $f : X \rightarrow Y$ a morphism. Prove that f is constant.

Exercise 34. Show that any two projective curves (i.e., hypersurfaces) in \mathbb{P}^2 have a non-empty intersection. (Hint: first do [HAG], Exercise I.3.5.)

Exercise 35. (Elimination theory) Translate Theorem I.5.7A of [HAG] into the following statement: let X be a projective variety, and let Y be any variety. The projection $p_Y : X \times Y \rightarrow Y$ is a closed map, i.e., it sends closed sets to closed sets. (Hint: first do the case $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.)

A variety X which has $p_Y : X \times Y \rightarrow Y$ closed for all varieties Y is called *complete*. Is \mathbb{A}^1 a complete variety?

Exercise 36. Let $f : X \rightarrow Y$ be a morphism of varieties. We call $\Gamma_f = \{(p, q) \in X \times Y : q = f(p)\}$ the *graph* of f .

(i) Prove that Γ_f is a closed subset of $X \times Y$.

(ii) Prove that the image of a projective variety under a morphism is closed. (Use (i) and elimination theory.)

(iii) Use (ii) to prove that any regular function on a projective variety is constant (this is Theorem I.3.4(a) of [HAG]).

Exercise 37. (Rigidity lemma) Let X, Y and Z be varieties, with X projective. Let $f : X \times Y \rightarrow Z$ be a morphism. Suppose that there is a point $y_0 \in Y$ such that f is constant on $X \times \{y_0\}$. Then f factors through the projection $p_Y : X \times Y \rightarrow Y$, i.e., f is constant on every slice $X \times \{y\}$. (Hint: choose any point $x_0 \in X$, and define $g : Y \rightarrow Z$ by $g(y) = f(x_0, y)$. To prove that $f = g \cdot p_Y$, it is enough to show that they agree on an open dense subset of $X \times Y$. If U is any open affine neighbourhood of $z_0 = f(x_0, y_0)$, consider the set $W = p_Y(f^{-1}(Z \setminus U))$. Prove that W is closed in Y , using elimination theory. By construction, $y_0 \notin W$, so that $Y \setminus W$ is a dense open subset of Y . Use Exercise 33(ii) to show that $f(X \times \{y\})$ is a point for any $y \notin W$.)

Exercise 38. Let X, Y be group varieties (cf. [HAG], Exerc. I.3.21) with X projective and let $f : X \rightarrow Y$ be a morphism. Prove that f is a homomorphism followed by a translation, i.e., prove that there is an $y \in Y$ and a homomorphism $g : X \rightarrow Y$ such that $f(x) = g(x) + y$ for all $x \in X$. (Hint: after a translation we may assume that $f(e_X) = e_Y$. Rephrase the condition that f be a homomorphism in terms of the constancy of a certain map $X \times X \rightarrow Y$. Use the rigidity lemma to prove this constancy.)

As an application, prove that a projective group variety is commutative. A projective group variety is called an *abelian variety*.

Exercise 39. Let k be a field and let K be an extension field of k . A subset S of K is called *algebraically dependent* over k if for some positive integer n there exists a non-zero polynomial f in $k[x_1, \dots, x_n]$ such that $f(s_1, \dots, s_n) = 0$ for some distinct $s_1, \dots, s_n \in S$. We call S *algebraically independent* over k if S is not algebraically dependent over k . A *transcendence base* of K over k is a subset of K which is algebraically independent over k and is maximal in the collection of all algebraically independent subsets of K .

(i) Prove that K has a transcendence base over k .

(ii) Let S be algebraically independent over k , and let $t \in K \setminus k(S)$. Prove: $S \cup \{t\}$ is algebraically independent over k iff t is transcendental over $k(S)$.

(iii) Again let S be algebraically independent over k . Prove: S is a transcendence base over k iff K is algebraic over $k(S)$.

(iv) Prove: if S is a finite transcendence base of K over k , then every transcendence base of K over k has the same number of elements as S . We call this number the *transcendence degree* of K over k , notation $\text{trdeg}_k K$.

Exercise 40. Show how the geometric version of Krull's Hauptidealsatz given in class follows from the algebraic version given in [HAG], Theorem I.1.11A.

Exercise 41. Prove that a group variety is non-singular.

Exercise 42. Compute the tangent lines to the conic $Z(x_0^2 + x_1^2 - 2x_2^2) \subseteq \mathbb{P}_{\mathbb{C}}^2$ that pass through $(0 : 0 : 1)$. Make it clear by means of a picture that these tangent lines can not be defined over \mathbb{R} .

Exercise 43. Compute the singularities of the Klein quartic curve $Z(x^3y + y^3z + z^3x) \subseteq \mathbb{P}^2$. (Note: the ground field may have any characteristic!)

Exercise 44. Compute the strict transforms in $\text{Bl}_0(\mathbb{A}^2)$ of the plane curves given in [HAG], Exercise I.5.1.

Exercise 45. Let X be an affine variety and let $p \in X$ be a point. Prove that the elements in $T_{X,p}$ correspond 1-1 with the k -algebra homomorphisms $\phi : A(X) \rightarrow k[\epsilon]/(\epsilon^2)$ with $\phi(\mathfrak{m}_p) \subseteq (\epsilon)$.

Exercise 46. If $m > n$, prove that there are no non-constant morphisms $\mathbb{P}^m \rightarrow \mathbb{P}^n$.

Exercise 47. Let k be an algebraically closed field. Compute the Hilbert function and polynomial for the ring

$$k[x, y, z, w]/(x, y) \cap (z, w)$$

corresponding to the disjoint union of two lines in \mathbb{P}^3 . Compare these to the Hilbert function and polynomial of the ring corresponding to one projective line.

Exercise 48. Compute the Hilbert function and polynomial for the twisted cubic curve, cf. [HAG], Exercise I.2.9(b).