Additional exercises Topics in Geometry, Fall 2007
Rings are always supposed to be commutative with 1.

Exercise 1. Let k be a field and let n be a positive integer. The *ring of polynomial functions* $O(k^n)$ on k^n is the ring of functions $f : k^n \to k$ such that there exists a polynomial g in $k[x_1, \ldots, x_n]$ with $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in k^n$.
(i) Determine $O(k)$ for k a finite field with q elements.
(ii) Assume that k is an infinite field. Prove that $O(k^n) \cong k[x_1, \ldots, x_n]$.

Exercise 2. Let A be a ring and let I be an ideal of A. The *radical* $\sqrt{I}$ of I is defined as

$$\sqrt{I} = \{a \in A \,:\, \text{there exists}\, n > 0\, \text{such that}\, a^n \in I\}.$$

Prove:
(i) $\sqrt{I} \supseteq I$;
(ii) $\sqrt{I}$ is an ideal of A;
(iii) $\sqrt{\sqrt{I}} = \sqrt{I}$;
(iv) $\sqrt{I}$ equals the intersection of all prime ideals containing I. (Hint: in order to prove the difficult inclusion, take an arbitrary $a \notin \sqrt{I}$. We want the existence of a prime ideal $\mathfrak{p} \supseteq I$ such that $a \notin \mathfrak{p}$. Consider the collection of ideals J of A that contain I but have $a \notin \sqrt{J}$. Apply Zorn's Lemma to this (clearly non-empty) collection. Any maximal element of it will be a prime ideal.)

Exercise 3. Determine the irreducible components of $Z(y^4 - x^6, y^3 - xy^2 - yx^3 + x^4)$ in $\mathbb{A}^2$.

Exercise 4. This exercise is devoted to a proof of the Cayley-Hamilton theorem: let k be a field, and let $M_n(k)$ be the set of n-by-n matrices with coefficients in k. For each $M \in M_n(k)$ denote by $\phi_M = \det(T \cdot \mathrm{Id} - M) \in k[T]$ the characteristic polynomial of M. Then $\phi_M(M) = 0$. Note that we may assume that k is algebraically closed.
(i) Give a natural identification of $M_n(k)$ with $\mathbb{A}_k^{n^2}$, thus giving $M_n(k)$ the structure of an affine variety.
(ii) Show that the set $CH := \{M \in M_n(k) \,:\, \phi_M(M) = 0\}$ is a Zariski closed subset of $M_n(k)$.
(iii) Show that the set $DEV := \{M \in M_n(k) \,:\, M\,\text{has no double eigenvalues}\}$ is non-empty and Zariski open in $M_n(k)$.
(iv) Show that $DEV \subset CH$. (Hint: CH contains all diagonal matrices).
(v) Prove that $CH = M_n(k)$.

Exercise 5. Suppose I is an ideal of a ring R. Show that if $\sqrt{I}$ is finitely generated, then for some integer N we have $\sqrt{I}^N \subseteq I$. Conclude that in a noetherian ring, the ideals I and J have the same radical iff there is some integer N such that $I^N \subseteq J$ and $J^N \subseteq I$. Use Hilbert's Nullstellensatz to deduce that if $I, J \subseteq A = k[x_1, \ldots, x_n]$ are ideals and k is an algebraically closed field, then $Z(I) = Z(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some integer N.

Exercise 6. (Integrality) Let A be a subring of a ring B. We call an element $b \in B$ *integral* over A if there exist $a_0, \ldots, a_{m-1} \in A$ such that $b^m + a_{m-1}b^{m-1} + \cdots + a_0 = 0$. We call B integral over A if every element of B is integral over A.
(i) Let $b \in B$. Prove that the following statements are equivalent: (1) b is integral over A; (2)

$A[b]$ is a finitely generated A-module; (3) $A[b]$ is contained in a subring C of B such that C is a finitely generated A-module. (Hint: in order to prove (3) $\Rightarrow$ (1), write $C = Af_1 + \cdots + Af_n$. For each i there are $c_{ij} \in A$ such that $b \cdot f_i = \sum_j c_{ij} \cdot f_j$. In other words, the vector ${}^t(f_1, \ldots, f_n)$ is in the kernel of the matrix $m_{ij} := b \cdot \delta_{ij} - c_{ij}$. Use the adjoint matrix of m_{ij} to get from this that $d \cdot f_i = 0$ for all i, where d is the determinant of m_{ij}. Since 1 is a linear combination of the f_i, we get then that $d = 0$. Finish by observing that d is a monic polynomial in b with coefficients in A.)
(ii) Let $b_1, \ldots, b_n$ be elements of B, each integral over A. Prove that the ring $A[b_1, \ldots, b_n]$ is a finitely generated A-module.
(iii) Prove that the set of elements of B which are integral over A is a subring of B containing A.
(iv) Let $A \subseteq B \subseteq C$ be rings, and assume B integral over A, and C integral over B. Prove that C is integral over A.
(v) Let B be a domain and let A be a subring of B such that B is integral over A. Prove: B is a field $\Leftrightarrow$ A is a field.

Exercise 7. Describe the maximal ideals of $k[x]$ for various fields k such as $k = \mathbb{R}$, k a finite field, ...

Exercise 8. Let X be an affine variety. For f in $A(X)$ define $D(f) = \{p \in X : f(p) \neq 0\}$ and $Z(f) = X \setminus D(f) = \{p \in X : f(p) = 0\}$. Prove:
(i) $X = \cup_{i \in I} D(f_i) \Leftrightarrow \cap_{i \in I} Z(f_i) = \varnothing \Leftrightarrow$ the f_i generate $A(X)$ as an ideal.
(ii) The $D(f)$ with f running through $A(X)$ form a basis for the Zariski topology on X.
(iii) X is quasi-compact, i.e., every covering of X with open subsets has a finite subcovering.

Exercise 9. Let S be a graded ring. An ideal $\mathfrak{a} \subseteq S$ is called *homogeneous* if every $f \in \mathfrak{a}$ has all its components in $\mathfrak{a}$. Prove:
(i) An ideal is homogeneous iff it can be generated by homogeneous elements.
(ii) The sum, product, intersection and radical of homogeneous ideals are homogeneous.
(iii) The homogeneous ideal $\mathfrak{a}$ is prime iff for any two homogeneous elements $f, g \in S$ one has that $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

Exercise 10. Let k be a field, let n be a positive integer and let $\mathbb{P}^n$ be projective n-space over k. Prove that there is a decomposition

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \ldots \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0$$

in disjoint subsets. Compute the number of elements of $\mathbb{P}^n$ for k a finite field of q elements.

Exercise 11. For any $d \in \mathbb{Z}_{\geq 0}$, let $S_d \subseteq S = k[x_0, \ldots, x_n]$ be the k-vector space of homogeneous polynomials of degree d. Prove that $\dim_k S_d = \binom{d+n}{n}$.

Exercise 12. (Localisation) Let A be a ring and let $S \subseteq A$ be a multiplicative subset, i.e., a subset S of A with $1 \in S$ and closed under multiplication. Define a relation $\equiv$ on $A \times S$ as follows: let $(a, s) \equiv (b, t) \Leftrightarrow u \cdot (at - bs) = 0$ for some $u \in S$.
(i) Prove that $\equiv$ is an equivalence relation.
We denote by $S^{-1}A$ the set of equivalence classes for $\equiv$.

(ii) Prove that $S^{-1}A$ has a natural ring structure, and that there is a natural ring homomorphism $\phi : A \to S^{-1}A$.
For $f \in A$ one usually denotes $S^{-1}A$ for $S = \{1\} \cup \{f^n\}_{n \in \mathbb{Z}_{>0}}$ by A_f. For example, if $A = \mathbb{Z}$ and $f = 2$ then A_f is the ring of rational numbers a/b with $a, b \in \mathbb{Z}$ such that b is a power of 2.
(iii) Prove that A_f is the zero ring $\Leftrightarrow f$ is nilpotent. Hence the map ϕ of (ii) need not be injective.
For $\mathfrak{p}$ a prime ideal of A one usually denotes $S^{-1}A$ for $S = A - \mathfrak{p}$ by $A_\mathfrak{p}$. For example, if $A = \mathbb{Z}$ and $\mathfrak{p} = (3)$ then $A_\mathfrak{p}$ is the ring of rational numbers a/b with $a, b \in \mathbb{Z}$ such that b is not divisible by 3.
(iv) Prove that $A_\mathfrak{p}$ has a unique maximal ideal.
A ring with a unique maximal ideal is called a *local ring.*
(v) Prove that $S^{-1}A$ and ϕ satisfy the following universal property: let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for all s in S. Then there is a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \cdot \phi$.
(vi) Prove that if A is a subring of a ring B, and B is integral over A, then $S^{-1}B$ is integral over $S^{-1}A$.

Exercise 13. If A is a local ring with maximal ideal $\mathfrak{m}$, we call the field $A/\mathfrak{m}$ the *residue field* of A.
(i) If B is a ring and $\mathfrak{p}$ is a prime ideal of B, prove that the residue field of $B_\mathfrak{p}$ is isomorphic to the field of fractions of the domain $B/\mathfrak{p}$.
(ii) Prove that a ring A is local iff the set of non-units of A is an ideal of A.
(iii) Prove that a local ring is not a direct sum of two rings.

Exercise 14. Let X be an affine variety. For non-constant f in $A(X)$ let $Y = D(f)$ be the corresponding quasi-affine variety. Prove that Y is (naturally isomorphic to) an affine variety, and describe the coordinate ring of Y. (Use localisation. See also Lemma I.4.2 in [HAG].)

Exercise 15. Let k be an algebraically closed field and let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. Let g in $k[x_1, \ldots, x_n]$ be such that $f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0 \Rightarrow g(x_1, \ldots, x_n) = 0$. Rephrase our proof that $g \in \sqrt{(f_1, \ldots, f_r)}$ (the Hilbert Nullstellensatz!) as follows.
(i) Let $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and consider the localisation map $A \to A_{\bar{g}}$. Use the Weak Nullstellensatz to prove that $A_{\bar{g}}$ is the zero ring.
(ii) Conclude that $\bar{g}$ is nilpotent in A.
(iii) Conclude that $g \in \sqrt{(f_1, \ldots, f_r)}$.

Exercise 16. Prove that $X = Z(x^2 - y, y^2 - z)$ in $\mathbb{A}^3$ is irreducible, and that $X \cong \mathbb{A}^1$. Prove that $Z(x^2 - y^3) \subseteq \mathbb{A}^2$ is not isomorphic with $\mathbb{A}^1$. Give a bijective morphism from $\mathbb{A}^1$ to $Z(x^2 - y^3)$.

Exercise 17. Let k be an algebraically closed field. Describe the prime ideals of $k[x, y]$. Describe the algebraic subsets of $\mathbb{A}^2$.

Exercise 18. Let k be an algebraically closed field of characteristic $p > 0$, and let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ be polynomials with coefficients in a finite field $\mathbb{F}_q$. Let $X = Z(f_1, \ldots, f_r) \subseteq \mathbb{A}^n$.
(i) Show that the map $F_q : X \to X$ given by $(x_1, \ldots, x_n) \mapsto (x_1^q, \ldots, x_n^q)$ is a morphism. We

call this map the q-th *Frobenius map*.
(ii) Show that F_q is bijective.
(iii) Show that F_q is not an isomorphism unless X is a point.
(iv) Describe the fixed point set of F_q and prove that it is finite.

Exercise 19. Let k be an algebraically closed field. Prove that $PGL(2,k)$ acts via automorphisms on $\mathbb{P}^1$, and that the action is 3-transitive. Prove also that if a projective transformation of $\mathbb{P}^1$ has three fix points, then it is the identity.

Exercise 20. More generally, prove that $PGL(n+1,k)$ acts via automorphisms on $\mathbb{P}^n$.
We call a collection of $k \geq n+2$ distinct points in $\mathbb{P}^n$ to be *in general position* if no $n+1$ among them lie in a hyperplane. Let $p_0, \ldots, p_{n+1}$ be $n+2$ points in general position in $\mathbb{P}^n$. Prove that we can choose homogeneous coordinates such that $p_0 = (1:0:\ldots:0)$, $p_1 = (0:1:0:\ldots:0), \ldots, p_n = (0:\ldots:0:1)$ and $p_{n+1} = (1:1:\ldots:1)$.
Choose five points in general position in $\mathbb{P}^2$. Show that there is a unique non-degenerate conic passing through them.

Exercise 21. (Classification of conics over $\mathbb{C}$) Recall that a conic in the affine real plane $\mathbb{R}^2$ (that is, the locus in $\mathbb{R}^2$ defined by a quadratic equation with real coefficients) belongs to one of the following eight types:
(a) the empty set (as with $x^2+y^2+1=0$);
(b) a single point (as with $x^2+y^2=0$);
(c) a 'double line' ($x^2=0$);
(d) the union of two incident lines ($xy=0$);
(e) the union of two parallel lines ($x(x-1)=0$);
(f) a parabola ($y-x^2=0$);
(g) a hyperbola ($xy-1=0$);
(h) an ellipse ($x^2+2y^2-1=0$).
Any two examples of one of these types differ only by a (real) projective linear transformation.
(i) Show that in the affine complex plane $\mathbb{A}^2_{\mathbb{C}}$ there are only five types of conics: types (a) and (b) disappear, and types (g) and (h) coincide.
(ii) Show that in the projective complex plane $\mathbb{P}^2_{\mathbb{C}}$ there are only three types of conics: they are represented by types (c), (d) and (h) from the above list. (Hint: this is a classification by the *rank* of a conic, where the rank of a quadratic form $\sum_i a_{ii}x_i^2 + 2\sum_{i<j} a_{ij}x_ix_j$ is defined by the rank of the symmetric matrix (a_{ij}).)
(iii) Show that the different types in (i) correspond to the relative position of the conic and the line at infinity. More precisely, a parabola is a rank-3 conic tangent to the line at infinity, while an ellipse/hyperbola is a rank-3 conic meeting the line at infinity in two distinct points.
(iv) For a real conic C, complex conjugation acts in a natural way on the set of its complex-valued points. Assume that C belongs to type (g) or (h). Prove: C is a hyperbola iff its points at infinity are fixed under complex conjugation (i.e., real), and: C is an ellipse iff its points at infinity are conjugate.

Exercise 22. Let $A = (a_0 : a_1 : a_2)$ and $B = (b_0 : b_1 : b_2)$ be distinct points in $\mathbb{P}^2$. Give an equation for the line AB passing through A and B.

Exercise 23. (Pascal's Theorem) Let C be a rank-3 conic in $\mathbb{P}^2$, and let A, B, C, A', B', C' be six distinct points on C. Prove that the points $P = AB' \cap A'B$, $Q = AC' \cap A'C$ and $R = BC' \cap B'C$ are collinear.

Exercise 24. Let V be the projective variety of conics passing through four given points in general position in $\mathbb{P}^2$. Assume that $\mathrm{char} k \neq 2$. Show that the degenerate conics correspond to a proper algebraic subset of V, consisting of three points.

Exercise 25. Give an isomorphism between $\mathbb{P}^1$ and $Z(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$. Parametrise all integer solutions to the equation $x^2 + y^2 = z^2$.

Exercise 26. (Noetherian modules) Let A be a ring and let M be an A-module.
(i) Prove that the following conditions on M are equivalent: every submodule of M is finitely generated; every ascending chain $M_1 \subseteq M_2 \subseteq \ldots$ of submodules of M becomes stationary; every non-empty set of submodules of M has a maximal element. A module satisfying these conditions is called *noetherian.*
(ii) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Prove: M is noetherian iff M' and M'' are noetherian.
(iii) If M_i for $i = 1, \ldots, n$ are noetherian A-modules, then so is $\oplus_{i=1}^n M_i$. Prove this.
(iv) Assume that A is a noetherian ring and that M is a finitely generated A-module. Prove that M is noetherian.

Exercise 27. (Closedness can be checked locally) Let X be a topological space and $Y \subseteq X$. Prove: Y is closed in X $\Leftrightarrow$ there is a covering $X = \cup_i U_i$ with open subsets such that $Y \cap U_i$ is closed in U_i for all i $\Leftrightarrow$ for any covering $X = \cup_i U_i$ with open subsets one has $Y \cap U_i$ closed in U_i for all i.

Exercise 28. Let $(A, \mathfrak{m})$ be a local ring and let M be a finitely generated A-module. Prove the following statements, all known as *Nakayama's Lemma*:
(i) If $\mathfrak{m}M = M$, then $M = 0$.
(ii) Let N be a submodule of M. If $M = \mathfrak{m}M + N$ then $M = N$.
(iii) Let $x_1, \ldots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then the x_i generate M.

Exercise 29. Let $(A, \mathfrak{m})$ be a noetherian local domain with residue field k. Prove that the following conditions on A are equivalent:
(i) A is a euclidean domain;
(ii) A is a principal ideal domain;
(iii) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.
If these conditions are satisfied, A is called a *discrete valuation ring.*

Exercise 30. Let Y be a variety over an algebraically closed field k, let $U \subseteq Y$ be a non-empty open subset and $p \in U$ a point. Prove that the inclusion $U \hookrightarrow Y$ gives rise to natural isomorphisms $O_{p,U} \cong O_{p,Y}$ and $K(U) \cong K(Y)$ of k-algebras.

Exercise 31. Consider the "folium of Descartes" $X = Z(xyz - x^3 - y^3) \subseteq \mathbb{P}^2$. Prove that X is birationally equivalent with $\mathbb{P}^1$. Are they isomorphic?

Exercise 32. Consider the quadric $X = Z(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$. Prove that X is birationally equivalent with $\mathbb{A}^2$. Are they isomorphic?

Exercise 33. Prove that $\mathbb{P}^2$ is birationally equivalent with $\mathbb{P}^1 \times \mathbb{P}^1$. Are they isomorphic?

Exercise 34. (i) Let X be a variety that is both projective and affine. Prove that X consists of a single point.
(ii) Let X be a projective variety, Y an affine variety, and $f : X \to Y$ a morphism. Prove that f is constant.

Exercise 35. Show that any two projective curves (i.e., hypersurfaces) in $\mathbb{P}^2$ have a non-empty intersection. (Hint: first do [HAG], Exercise I.3.5.)

Exercise 36. (Elimination theory) Translate Theorem I.5.7A of [HAG] into the following statement: let X be a projective variety, and let Y be any variety. The projection $p_Y : X \times Y \to Y$ is a closed map, i.e., it sends closed sets to closed sets. (Hint: first do the case $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.)
A variety X which has $p_Y : X \times Y \to Y$ closed for all varieties Y is called *complete*. Is $\mathbb{A}^1$ a complete variety?

Exercise 37. Let $f : X \to Y$ be a morphism of varieties. We call $\Gamma_f = \{(p, q) \in X \times Y : q = f(p)\}$ the *graph* of f.
(i) Prove that Γ_f is a closed subset of $X \times Y$.
(ii) Prove that the image of a projective variety under a morphism is closed. (Use (i) and elimination theory.)
(iii) Use (ii) to prove that any regular function on a projective variety is constant (this is Theorem I.3.4(a) of [HAG]).

Exercise 38. (Rigidity lemma) Let X, Y and Z be varieties, with X projective. Let $f : X \times Y \to Z$ be a morphism. Suppose that there is a point $y_0 \in Y$ such that f is constant on $X \times \{y_0\}$. Then f factors through the projection $p_Y : X \times Y \to Y$, i.e., f is constant on every slice $X \times \{y\}$. (Hint: choose any point $x_0 \in X$, and define $g : Y \to Z$ by $g(y) = f(x_0, y)$. To prove that $f = g \cdot p_Y$, it is enough to show that they agree on an open dense subset of $X \times Y$. If U is any open affine neighbourhood of $z_0 = f(x_0, y_0)$, consider the set $W = p_Y(f^{-1}(Z \setminus U))$. Prove that W is closed in Y, using elimination theory. By construction, $y_0 \notin W$, so that $Y \setminus W$ is a dense open subset of Y. Use Exercise 34(ii) to show that $f(X \times \{y\})$ is a point for any $y \notin W$.)

Exercise 39. Let X, Y be group varieties (cf. [HAG], Exerc. I.3.21) with X projective and let $f : X \to Y$ be a morphism. Prove that f is a homomorphism followed by a translation, i.e., prove that there is an $y \in Y$ and a homomorphism $g : X \to Y$ such that $f(x) = g(x) + y$ for all $x \in X$. (Hint: after a translation we may assume that $f(e_X) = e_Y$. Rephrase the condition that f be a homomorphism in terms of the constancy of a certain map $X \times X \to Y$. Use the rigidity lemma to prove this constancy.)
As an application, prove that a projective group variety is commutative. A projective group variety is called an *abelian variety*.

Exercise 40. Let k be a field and let K be an extension field of k. A subset S of K is called *algebraically dependent* over k if for some positive integer n there exists a non-zero polynomial f in $k[x_1, \dots, x_n]$ such that $f(s_1, \dots, s_n) = 0$ for some distinct $s_1, \dots, s_n \in S$. We call S *algebraically independent* over k if S is not algebraically dependent over k. A *transcendence base* of K over k is a subset of K which is algebraically independent over k and is maximal in the collection of all algebraically independent subsets of K.
(i) Prove that K has a transcendence base over k.
(ii) Let S be algebraically independent over k, and let $t \in K \setminus k(S)$. Prove: $S \cup \{t\}$ is algebraically independent over k iff t is transcendent over $k(S)$.
(iii) Again let S be algebraically independent over k. Prove: S is a transcendence base over k iff K is algebraic over $k(S)$.
(iv) Prove: if S is a finite transcendence base of K over k, then every transcendence base of K over k has the same number of elements as S. We call this number the *transcendence degree* of K over k, notation $\mathrm{trdeg}_k K$.

Exercise 41. Show how the geometric version of Krull's Hauptidealsatz given in class follows from the algebraic version given in [HAG], Theorem I.1.11A.

Exercise 42. Prove that a group variety is non-singular.

Exercise 43. Compute the tangent lines to the conic $Z(x_0^2+x_1^2-2x_2^2) \subseteq \mathbb{P}^2_{\mathbb{C}}$ that pass through $(0 : 0 : 1)$. Make it clear by means of a picture that these tangent lines can not be defined over $\mathbb{R}$.

Exercise 44. Compute the singularities of the Klein quartic curve $Z(x^3y + y^3z + z^3x) \subseteq \mathbb{P}^2$. (Note: the ground field may have any characteristic!)

Exercise 45. Compute the strict transforms in $\mathrm{Bl}_0(\mathbb{A}^2)$ of the plane curves given in [HAG], Exercise I.5.1.

Exercise 46. Let X be an affine variety and let $p \in X$ be a point. Prove that the elements in $T_{X,p}$ correspond 1-1 with the k-algebra homomorphisms $\phi : A(X) \to k[\epsilon]/(\epsilon^2)$ with $\phi(\mathfrak{m}_p) \subseteq (\epsilon)$.

Exercise 47. If $m > n$, prove that there are no non-constant morphisms $\mathbb{P}^m \to \mathbb{P}^n$.

Exercise 48. Let k be an algebraically closed field. Compute the Hilbert function and polynomial for the ring

$$k[x, y, z, w]/(x, y) \cap (z, w)$$

corresponding to the disjoint union of two lines in $\mathbb{P}^3$. Compare these to the Hilbert function and polynomial of the ring corresponding to one projective line.

Exercise 49. Compute the Hilbert function and polynomial for the twisted cubic curve, cf. [HAG], Exercise I.2.9(b).