Exercise 1. Let $k$ be a field and let $n$ be a positive integer. The ring of polynomial functions $O(k^n)$ on $k^n$ is the ring of functions $f : k^n \to k$ such that there exists a polynomial $g$ in $k[x_1, \ldots, x_n]$ with $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in k^n$.

(i) Determine $O(k)$ for $k$ a finite field with $q$ elements.
(ii) Assume that $k$ is an infinite field. Prove that $O(k^n) \cong k[x_1, \ldots, x_n]$.

Exercise 2. Let $A$ be a ring and let $I$ be an ideal of $A$. The radical \(\sqrt{I}\) of $I$ is defined as

\[
\sqrt{I} = \{a \in A : \text{there exists } n > 0 \text{ such that } a^n \in I\}.
\]

Prove:
(i) $\sqrt{I} \supseteq I$;
(ii) $\sqrt{I}$ is an ideal of $A$;
(iii) $\sqrt{\sqrt{I}} = \sqrt{I}$;
(iv) $\sqrt{I}$ equals the intersection of all prime ideals containing $I$. (Hint: in order to prove the difficult inclusion, take an arbitrary $a \notin \sqrt{I}$. We want the existence of a prime ideal $p \supseteq I$ such that $a \notin p$. Consider the collection of ideals $J$ of $A$ that contain $I$ but have $a \notin \sqrt{J}$. Apply Zorn’s Lemma to this (clearly non-empty) collection. Any maximal element of it will be a prime ideal.)

Exercise 3. Determine the irreducible components of $Z(y^4 - x^6, y^3 - xy^2 - yx^3 + x^4)$ in $\mathbb{A}^2$.

Exercise 4. This exercise is devoted to a proof of the Cayley-Hamilton theorem: let $k$ be a field, and let $M_n(k)$ be the set of $n$-by-$n$ matrices with coefficients in $k$. For each $M \in M_n(k)$ denote by $\phi_M = \det(T \cdot \text{Id} - M) \in k[T]$ the characteristic polynomial of $M$. Then $\phi_M(M) = 0$. Note that we may assume that $k$ is algebraically closed.

(i) Give a natural identification of $M_n(k)$ with $\mathbb{A}^{n^2}_k$, thus giving $M_n(k)$ the structure of an affine variety.
(ii) Show that the set $CH := \{M \in M_n(k) : \phi_M(M) = 0\}$ is a Zariski closed subset of $M_n(k)$.
(iii) Show that the set $DEV := \{M \in M_n(k) : M$ has no double eigenvalues$\}$ is non-empty and Zariski open in $M_n(k)$.
(iv) Show that $DEV \subseteq CH$. (Hint: $CH$ contains all diagonal matrices).
(v) Prove that $CH = M_n(k)$.

Exercise 5. Suppose $I$ is an ideal of a ring $R$. Show that if $\sqrt{I}$ is finitely generated, then for some integer $N$ we have $\sqrt{I}^N \subseteq I$. Conclude that in a noetherian ring, the ideals $I$ and $J$ have the same radical iff there is some integer $N$ such that $I^N \subseteq J$ and $J^N \subseteq I$. Use Hilbert’s Nullstellensatz to deduce that if $I, J \subseteq A = k[x_1, \ldots, x_n]$ are ideals and $k$ is an algebraically closed field, then $Z(I) = Z(J)$ iff $I^N \subseteq J$ and $J^N \subseteq I$ for some integer $N$.

Exercise 6. (Integrality) Let $A$ be a subring of a ring $B$. We call an element $b \in B$ integral over $A$ if there exist $a_0, \ldots, a_{m-1} \in A$ such that $b^m + a_{m-1}b^{m-1} + \cdots + a_0 = 0$. We call $B$ integral over $A$ if every element of $B$ is integral over $A$.

(i) Let $b \in B$. Prove that the following statements are equivalent: (1) $b$ is integral over $A$; (2)
Exercise 9. Let \( P_k \) be a projective \( k \)-space. Compute the number of elements of \( P_k \) over \( a \).

(ii) Let \( b_1, \ldots, b_n \) be elements of \( B \), each integral over \( A \). Prove that the ring \( A[b_1, \ldots, b_n] \) is a finitely generated \( A \)-module.

(iii) Prove that the set of elements of \( B \) which are integral over \( A \) is a subring of \( B \) containing \( A \).

(iv) Let \( A \subseteq B \subseteq C \) be rings, and assume \( B \) integral over \( A \), and \( C \) integral over \( B \). Prove that \( C \) is integral over \( A \).

(v) Let \( B \) be a domain and let \( A \) be a subring of \( B \) such that \( B \) is integral over \( A \). Prove: \( B \) is a field \( \iff \) \( A \) is a field.

Exercise 7. Describe the maximal ideals of \( k[x] \) for various fields \( k \) such as \( k = \mathbb{R} \), \( k \) a finite field, ...

Exercise 8. Let \( X \) be an affine variety. For \( f \) in \( A(X) \) define \( D(f) = \{ p \in X : f(p) \neq 0 \} \) and \( Z(f) = X \setminus D(f) = \{ p \in X : f(p) = 0 \} \). Prove:

(i) \( X = \bigcup_{i \in I} D(f_i) \iff \bigcap_{i \in I} Z(f_i) = \emptyset \iff \{ f_i \} \) generate \( A(X) \) as an ideal.

(ii) The \( D(f) \) with \( f \) running through \( A(X) \) form a basis for the Zariski topology on \( X \).

(iii) \( X \) is quasi-compact, i.e., every covering of \( X \) with open subsets has a finite subcovering.

Exercise 9. Let \( S \) be a graded ring. An ideal \( \mathfrak{a} \subseteq S \) is called homogeneous if every \( f \in \mathfrak{a} \) has all its components in \( \mathfrak{a} \). Prove:

(i) An ideal is homogeneous iff it can be generated by homogeneous elements.

(ii) The sum, product, intersection and radical of homogeneous ideals are homogeneous.

(iii) The homogeneous ideal \( \mathfrak{a} \) is prime iff for any two homogeneous elements \( f, g \in S \) one has that \( fg \in \mathfrak{a} \) implies \( f \in \mathfrak{a} \) or \( g \in \mathfrak{a} \).

Exercise 10. Let \( k \) be a field, let \( n \) be a positive integer and let \( \mathbb{P}^n \) be projective \( n \)-space over \( k \). Prove that there is a decomposition

\[
\mathbb{P}^n = A^n \sqcup A^{n-1} \sqcup \cdots \sqcup A^1 \sqcup A^0
\]

in disjoint subsets. Compute the number of elements of \( \mathbb{P}^n \) for \( k \) a finite field of \( q \) elements.

Exercise 11. For any \( d \in \mathbb{Z}_{\geq 0} \), let \( S_d \subseteq S = k[x_0, \ldots, x_n] \) be the \( k \)-vector space of homogeneous polynomials of degree \( d \). Prove that \( \dim_k S_d = \binom{d+n}{n} \).

Exercise 12. (Localisation) Let \( A \) be a ring and let \( S \subseteq A \) be a multiplicative subset, i.e., a subset \( S \) of \( A \) with \( 1 \in S \) and closed under multiplication. Define a relation \( \equiv \) on \( A \times S \) as follows: let \( (a, s) \equiv (b, t) \iff u \cdot (at - bs) = 0 \) for some \( u \in S \).

(i) Prove that \( \equiv \) is an equivalence relation.

We denote by \( S^{-1}A \) the set of equivalence classes for \( \equiv \).
For $f \in A$ one usually denotes $S^{-1}A$ for $S = \{1\} \cup \{f^n\}_{n \in \mathbb{Z}_{>0}}$ by $A_f$. For example, if $A = \mathbb{Z}$ and $f = 2$ then $A_f$ is the ring of rational numbers $a/b$ with $a, b \in \mathbb{Z}$ such that $b$ is a power of 2.

(iii) Prove that $A_f$ is the zero ring $\iff f$ is nilpotent. Hence the map $\phi$ of (ii) need not be injective.

For $p$ a prime ideal of $A$ one usually denotes $S^{-1}A$ for $S = A - p$ by $A_p$. For example, if $A = \mathbb{Z}$ and $p = (3)$ then $A_p$ is the ring of rational numbers $a/b$ with $a, b \in \mathbb{Z}$ such that $b$ is not divisible by 3.

(iv) Prove that $A_p$ has a unique maximal ideal.

A ring with a unique maximal ideal is called a local ring.

(v) Prove that $S^{-1}A$ and $\phi$ satisfy the following universal property: let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in $B$ for all $s$ in $S$. Then there is a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \cdot \phi$.

(vi) Prove that if $A$ is a subring of a ring $B$, and $B$ is integral over $A$, then $S^{-1}B$ is integral over $S^{-1}A$.

Exercise 13. If $A$ is a local ring with maximal ideal $m$, we call the field $A/m$ the residue field of $A$.

(i) If $B$ is a ring and $p$ is a prime ideal of $B$, prove that the residue field of $B_p$ is isomorphic to the field of fractions of the domain $B/p$.

(ii) Prove that a ring $A$ is local iff the set of non-units of $A$ is an ideal of $A$.

(iii) Prove that a local ring is not a direct sum of two rings.

Exercise 14. Let $X$ be an affine variety. For non-constant $f$ in $A(X)$ let $Y = D(f)$ be the corresponding quasi-affine variety. Prove that $Y$ is (naturally isomorphic to) an affine variety, and describe the coordinate ring of $Y$. (Use localisation. See also Lemma I.4.2 in [HAG].)

Exercise 15. Let $k$ be an algebraically closed field and let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. Let $g$ in $k[x_1, \ldots, x_n]$ be such that $f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0 \Rightarrow g(x_1, \ldots, x_n) = 0$. Rephrase our proof that $g \in \sqrt{(f_1, \ldots, f_r)}$ (the Hilbert Nullstellensatz!) as follows.

(i) Let $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and consider the localisation map $A \to A_{\bar{g}}$. Use the Weak Nullstellensatz to prove that $A_{\bar{g}}$ is the zero ring.

(ii) Conclude that $\bar{g}$ is nilpotent in $A$.

(iii) Conclude that $g \in \sqrt{(f_1, \ldots, f_r)}$.

Exercise 16. Prove that $X = \mathbb{Z}(x^2 - y, y^2 - z)$ in $\mathbb{A}^3$ is irreducible, and that $X \cong \mathbb{A}^1$. Prove that $\mathbb{Z}(x^2 - y^3) \subseteq \mathbb{A}^2$ is not isomorphic with $\mathbb{A}^1$. Give a bijective morphism from $\mathbb{A}^1$ to $\mathbb{Z}(x^2 - y^3)$.

Exercise 17. Let $k$ be an algebraically closed field. Describe the prime ideals of $k[x, y]$. Describe the algebraic subsets of $\mathbb{A}^2$.

Exercise 18. Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ be polynomials with coefficients in a finite field $\mathbb{F}_q$. Let $X = \mathbb{Z}(f_1, \ldots, f_r) \subseteq \mathbb{A}^n$.

(i) Show that the map $F_q : X \to X$ given by $(x_1, \ldots, x_n) \mapsto (x_1^q, \ldots, x_n^q)$ is a morphism. We
call this map the $q$-th Frobenius map.
(ii) Show that $F_q$ is bijective.
(iii) Show that $F_q$ is not an isomorphism unless $X$ is a point.
(iv) Describe the fixed point set of $F_q$ and prove that it is finite.

Exercise 19. Let $k$ be an algebraically closed field. Prove that $PGL(2, k)$ acts via automorphisms on $\mathbb{P}^1$, and that the action is 3-transitive. Prove also that if a projective transformation of $\mathbb{P}^1$ has three fix points, then it is the identity.

Exercise 20. More generally, prove that $PGL(n + 1, k)$ acts via automorphisms on $\mathbb{P}^n$. We call a collection of $k \geq n + 2$ distinct points in $\mathbb{P}^n$ to be in general position if no $n + 1$ among them lie in a hyperplane. Let $p_0, \ldots, p_{n+1}$ be $n + 2$ points in general position in $\mathbb{P}^n$. Prove that we can choose homogeneous coordinates such that $p_0 = (1 : 0 : \ldots : 0)$, $p_1 = (0 : 1 : 0 : \ldots : 0), \ldots, p_n = (0 : \ldots : 0 : 1)$ and $p_{n+1} = (1 : 1 : \ldots : 1)$.

Choose five points in general position in $\mathbb{P}^2$. Show that there is a unique non-degenerate conic passing through them.

Exercise 21. (Classification of conics over $\mathbb{C}$) Recall that a conic in the affine real plane $\mathbb{R}^2$ (that is, the locus in $\mathbb{R}^2$ defined by a quadratic equation with real coefficients) belongs to one of the following eight types:
(a) the empty set (as with $x^2 + y^2 + 1 = 0$);
(b) a single point (as with $x^2 + y^2 = 0$);
(c) a ‘double line’ ($x^2 = 0$);
(d) the union of two incident lines ($xy = 0$);
(e) the union of two parallel lines ($xy = 0$);
(f) a parabola ($y - x^2 = 0$);
(g) a hyperbola ($xy - 1 = 0$);
(h) an ellipse ($x^2 + 2y^2 - 1 = 0$).
Any two examples of one of these types differ only by a (real) projective linear transformation.
(i) Show that in the affine complex plane $\mathbb{A}^2_\mathbb{C}$ there are only five types of conics: types (a) and (b) disappear, and types (g) and (h) coincide.
(ii) Show that in the projective complex plane $\mathbb{P}^2_\mathbb{C}$ there are only three types of conics: they are represented by types (c), (d) and (h) from the above list. (Hint: this is a classification by the rank of a conic, where the rank of a quadratic form $\sum_i a_{ii}x_i^2 + 2\sum_{i<j}a_{ij}x_ix_j$ is defined by the rank of the symmetric matrix $(a_{ij})$.)
(iii) Show that the different types in (i) correspond to the relative position of the conic and the line at infinity. More precisely, a parabola is a rank-3 conic tangent to the line at infinity, while an ellipse/hyperbola is a rank-3 conic meeting the line at infinity in two distinct points.
(iv) For a real conic $C$, complex conjugation acts in a natural way on the set of its complex-valued points. Assume that $C$ belongs to type (g) or (h). Prove: $C$ is a hyperbola iff its points at infinity are fixed under complex conjugation (i.e., real), and: $C$ is an ellipse iff its points at infinity are conjugate.

Exercise 22. Let $A = (a_0 : a_1 : a_2)$ and $B = (b_0 : b_1 : b_2)$ be distinct points in $\mathbb{P}^2$. Give an equation for the line $AB$ passing through $A$ and $B$. 

4
Exercise 23. (Pascal’s Theorem) Let $C$ be a rank-3 conic in $\mathbb{P}^2$, and let $A, B, C, A', B', C'$ be six distinct points on $C$. Prove that the points $P = AB' \cap A'B$, $Q = AC' \cap A'C$ and $R = BC' \cap B'C$ are collinear.

Exercise 24. Let $V$ be the projective variety of conics passing through four given points in general position in $\mathbb{P}^2$. Assume that $\text{char} k \neq 2$. Show that the degenerate conics correspond to a proper algebraic subset of $V$, consisting of three points.

Exercise 25. Give an isomorphism between $\mathbb{P}^1$ and $\mathbb{Z}(x^2 + y^2 - z^2) \subseteq \mathbb{P}^2$. Parametrise all integer solutions to the equation $x^2 + y^2 = z^2$.

Exercise 26. (Noetherian modules) Let $A$ be a ring and let $M$ be an $A$-module.
(i) Prove that the following conditions on $M$ are equivalent: every submodule of $M$ is finitely generated; every ascending chain $M_1 \subseteq M_2 \subseteq \ldots$ of submodules of $M$ becomes stationary; every non-empty set of submodules of $M$ has a maximal element. A module satisfying these conditions is called noetherian.
(ii) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $A$-modules. Prove: $M$ is noetherian iff $M'$ and $M''$ are noetherian.
(iii) If $M_i$ for $i = 1, \ldots, n$ are noetherian $A$-modules, then so is $\oplus_{i=1}^n M_i$. Prove this.
(iv) Assume that $A$ is a noetherian ring and that $M$ is a finitely generated $A$-module. Prove that $M$ is noetherian.

Exercise 27. (Closedness can be checked locally) Let $X$ be a topological space and $Y \subseteq X$. Prove: $Y$ is closed in $X$ $\iff$ there is a covering $X = \bigcup_i U_i$ with open subsets such that $Y \cap U_i$ is closed in $U_i$ for all $i$ $\iff$ for any covering $X = \bigcup_i U_i$ with open subsets one has $Y \cap U_i$ closed in $U_i$ for all $i$.

Exercise 28. Let $(A, m)$ be a local ring and let $M$ be a finitely generated $A$-module. Prove the following statements, all known as Nakayama’s Lemma:
(i) If $mM = M$, then $M = 0$.
(ii) Let $N$ be a submodule of $M$. If $M = mM + N$ then $N = M$.
(iii) Let $x_1, \ldots, x_n$ be elements of $M$ whose images in $M/mM$ form a basis of this vector space. Then the $x_i$ generate $M$.

Exercise 29. Let $(A, m)$ be a noetherian local domain with residue field $k$. Prove that the following conditions on $A$ are equivalent:
(i) $A$ is a euclidean domain;
(ii) $A$ is a principal ideal domain;
(iii) $\dim_k m/m^2 = 1$.
If these conditions are satisfied, $A$ is called a discrete valuation ring.

Exercise 30. Let $U, V$ be open affine subvarieties of a variety $X$. Prove that $U \cap V$ is again an affine variety.

Exercise 31. Consider the “folium of Descartes” $X = Z(xyz - x^3 - y^3) \subseteq \mathbb{P}^2$. Prove that $X$ is birationally equivalent with $\mathbb{P}^1$. Are they isomorphic?

Exercise 32. Consider the quadric $X = Z(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$. Prove that $X$ is birationally equivalent with $\mathbb{A}^2$. Are they isomorphic?
Exercise 33. Prove that \( \mathbb{P}^2 \) is birationally equivalent with \( \mathbb{P}^1 \times \mathbb{P}^1 \). Are they isomorphic?

Exercise 34. (i) Let \( X \) be a variety that is both projective and affine. Prove that \( X \) consists of a single point.
(ii) Let \( X \) be a projective variety, \( Y \) an affine variety, and \( f : X \to Y \) a morphism. Prove that \( f \) is constant.

Exercise 35. Let \( X \) be an affine variety and let \( X \to \mathbb{A}^n \) be a morphism. Is the image of \( X \) closed in \( \mathbb{A}^n \)?

Exercise 36. (Elimination theory) Translate Theorem I.5.7A of [HAG] into the following statement: let \( X \) be a projective variety, and let \( Y \) be any variety. The projection \( p_Y : X \times Y \to Y \) is a closed map, i.e., it sends closed sets to closed sets. (Hint: first do the case \( X = \mathbb{P}^n \) and \( Y = \mathbb{A}^m \).

A variety \( X \) which has \( p_Y : X \times Y \to Y \) closed for all varieties \( Y \) is called complete. Is \( \mathbb{A}^1 \) a complete variety?

Exercise 37. Let \( f : X \to Y \) be a morphism of varieties. We call \( \Gamma_f = \{(p, q) \in X \times Y : q = f(p)\} \) the graph of \( f \).
(i) Prove that \( \Gamma_f \) is a closed subset of \( X \times Y \).
(ii) Prove that the image of a projective variety under a morphism is closed. (Use (i) and elimination theory.)
(iii) Use (ii) to prove that any regular function on a projective variety is constant (this is Theorem I.3.4(a) of [HAG]).

Exercise 38. (Rigidity lemma) Let \( X, Y \) and \( Z \) be varieties, with \( X \) projective. Let \( f : X \times Y \to Z \) be a morphism. Suppose that there is a point \( y_0 \in Y \) such that \( f \) is constant on \( X \times \{y_0\} \). Then \( f \) factors through the projection \( p_Y : X \times Y \to Y \), i.e., \( f \) is constant on every slice \( X \times \{y\} \). (Hint: choose any point \( x_0 \in X \), and define \( g : Y \to Z \) by \( g(y) = f(x_0, y) \). To prove that \( f = g \cdot p_Y \), it is enough to show that they agree on an open dense subset of \( X \times Y \). If \( U \) is any open affine neighbourhood of \( z_0 = f(x_0, y_0) \), consider the set \( W = p_Y(f^{-1}(Z \setminus U)) \).

Prove that \( W \) is closed in \( Y \), using elimination theory. By construction, \( y_0 \notin W \), so that \( Y \setminus W \) is a dense open subset of \( Y \). Use Exercise 34(ii) to show that \( f(X \times \{y\}) \) is a point for any \( y \notin W \).

Exercise 39. Let \( X, Y \) be group varieties (cf. [HAG], Exerc. I.3.21) with \( X \) projective and let \( f : X \to Y \) be a morphism. Prove that \( f \) is a homomorphism followed by a translation, i.e., prove that there is an \( y \in Y \) and a homomorphism \( g : X \to Y \) such that \( f(x) = g(x) + y \) for all \( x \in X \). (Hint: after a translation we may assume that \( f(e_X) = e_Y \). Rephrase the condition that \( f \) be a homomorphism in terms of the constancy of a certain map \( X \times X \to Y \).

Use the rigidity lemma to prove this constancy.)

As an application, prove that a projective group variety is commutative. A projective group variety is called an abelian variety.

Exercise 40. Let \( k \) be a field and let \( K \) be an extension field of \( k \). A subset \( S \) of \( K \) is called algebraically dependent over \( k \) if for some positive integer \( n \) there exists a non-zero polynomial \( f \) in \( k[x_1, \ldots, x_n] \) such that \( f(s_1, \ldots, s_n) = 0 \) for some distinct \( s_1, \ldots, s_n \in S \).
We call \( S \) \textit{algebraically independent} over \( k \) if \( S \) is not algebraically dependent over \( k \). A \textit{transcendence base} of \( K \) over \( k \) is a subset of \( K \) which is algebraically independent over \( k \) and is maximal in the collection of all algebraically independent subsets of \( K \).

(i) Prove that \( K \) has a transcendence base over \( k \).

(ii) Let \( S \) be algebraically independent over \( k \), and let \( t \in K \setminus k(S) \). Prove: \( S \cup \{ t \} \) is algebraically independent over \( k \) iff \( t \) is transcendental over \( k \).

(iii) Again let \( S \) be algebraically independent over \( k \). Prove: \( S \) is a transcendence base over \( k \) iff \( K \) is algebraic over \( k \).

(iv) Prove: if \( S \) is a finite transcendence base of \( K \) over \( k \), then every transcendence base of \( K \) over \( k \) has the same number of elements as \( S \). We call this number the \textit{transcendence degree} of \( K \) over \( k \), notation \( \text{trdeg}_k K \).

\textbf{Exercise 41.} Show how the geometric version of Krull’s Hauptidealsatz given in class follows from the algebraic version given in [HAG], Theorem I.1.11A.

\textbf{Exercise 42.} Prove that a group variety (cf. [HAG], Exerc. I.3.21) is non-singular.

\textbf{Exercise 43.} Compute the tangent lines to the conic \( Z(x_0^2 + x_1^2 - 2x_2^2) \subseteq \mathbb{P}_C^2 \) that pass through \((0 : 0 : 1)\). Make it clear by means of a picture that these tangent lines can not be defined over \( \mathbb{R} \).

\textbf{Exercise 44.} Compute the singularities of the Klein quartic curve \( Z(x^3y + y^3z + z^3x) \subseteq \mathbb{P}^2 \).

(Note: the ground field may have any characteristic!)

\textbf{Exercise 45.} Compute the strict transforms in \( \text{Bl}_0(\mathbb{A}^2) \) of the plane curves given in [HAG], Exercise I.5.1.

\textbf{Exercise 46.} Let \( X \) be an affine variety and let \( p \in X \) be a point. Prove that the elements in \( T_{X,p} \) correspond 1-1 with the \( k \)-algebra homomorphisms \( \phi: A(X) \rightarrow k[\epsilon]/(\epsilon^2) \) with \( \phi(m_p) \subseteq (\epsilon) \).

\textbf{Exercise 47.} If \( m > n \), prove that there are no non-constant morphisms \( \mathbb{P}^m \rightarrow \mathbb{P}^n \).

\textbf{Exercise 48.} Let \( k \) be an algebraically closed field. Compute the Hilbert function and polynomial for the ring \( k[x, y, z, w]/(x, y) \cap (z, w) \) corresponding to the disjoint union of two lines in \( \mathbb{P}^3 \). Compare these to the Hilbert function and polynomial of the ring corresponding to one projective line.

\textbf{Exercise 49.} Compute the Hilbert function and polynomial for the twisted cubic curve, cf. [HAG], Exercise I.2.9(b).

\textbf{Exercise 50.} (Posed by Matthijs van Duin) Let \( X \) be a variety such that the natural inclusion \( Q(O(X)) \rightarrow K(X) \) is an isomorphism. Is \( X \) quasi-affine?