Two more exercises, Topics in Geometry I, Fall 2008

Exercise 1. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d > 1$. Let m be an integer such that $2m \geq n$. Prove that X contains no linear varieties of dimension m.

Exercise 2. *(The Grassmann variety of lines in $\mathbb{P}^3$)* Let G be the set of lines in $\mathbb{P}^3$. We define a map $\psi : G \to \mathbb{P}^5$ by sending a line given by two linear equations:

$$\begin{aligned} a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 &= 0\,, \\ b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 &= 0 \end{aligned}$$

to the point:

$$(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5$$

given by:

$$p_{01} = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}, \, p_{02} = \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix}, \ldots, p_{23} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

Prove the following statements.
(i) The map ψ is well-defined and induces a bijection between G and the points of the quadric hypersurface $Q \subset \mathbb{P}^5$ given by:

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0\,.$$

(ii) The quadric Q is non-singular.
(iii) The lines contained in a hyperplane $H \subset \mathbb{P}^3$ are, via ψ, in bijection with a projective plane contained in Q.
(iv) Let L be a line in $\mathbb{P}^3$ and let T_LQ be the tangent space to Q at the point corresponding to L. Then the set of lines intersecting L is, via ψ, in bijection with $T_LQ \cap Q$.
(v) Let L_0, L_1, L_2, L_3 be four lines in $\mathbb{P}^3$. Prove that there exist lines in $\mathbb{P}^3$ that pass through them all. Prove that if the number of lines that pass through L_0, L_1, L_2, L_3 is finite, then this number is at most 2.