

Exercises Topics in Geometry - Sheaves and sheaf cohomology, Fall 2009

Exercise 1. Consider the following operations on groups G :

- forming the commutator subgroup $[G, G]$ of G ;
- forming the abelianization $G^{\text{ab}} = G/[G, G]$ of G ;
- forming the center $Z(G)$ of G ;
- taking $\text{Hom}(H, G)$ where H is a fixed group;
- taking $\text{Hom}(G, H)$ where H is a fixed group;
- forming the set of conjugacy classes of G ;
- forming the automorphism group $\text{Aut}(G)$.

Which of the above operations naturally give rise to a functor on the category of groups? Motivate your answer.

Exercise 2. Consider the following property (*) for an abelian group A :

for every inclusion $I \subset \mathbb{Z}$ of a subgroup I in \mathbb{Z} , and every homomorphism $f: I \rightarrow A$, there exists a homomorphism $g: \mathbb{Z} \rightarrow A$ such that $g|_I = f$.

(i) Verify that saying that A satisfies (*) is equivalent to saying that A is divisible.

(ii) Prove that if A satisfies (*), then A is injective.

Hint: let $M \subset N$ be an inclusion of abelian groups, and let $k: M \rightarrow A$ be a homomorphism. Consider the set of pairs (H, h) where H is a subgroup of N with $M \subset H$ and where $h: H \rightarrow A$ is a homomorphism with $h|_M = k$. This set has a natural partial ordering. Prove that a maximal element of this set is of the form (N, h) , and verify that such a maximal element exists by Zorn's Lemma.

(iii) Conclude that an abelian group A is injective if and only if A is divisible.

Exercise 3. Prove that the category of abelian groups has enough injectives.

Hint: for each abelian group A there exists a free abelian group F and a surjective morphism $F \rightarrow A$. As F is a direct sum of copies of \mathbb{Z} , the group F can be embedded in a divisible group. Furthermore, a quotient of a divisible group is divisible.

Exercise 4. Let Ab be the category of abelian groups, and let A be a fixed abelian group.

(i) Show that the rule $M \mapsto \text{Hom}(A, M)$ defines a left exact functor from Ab to itself.

Let B be a cyclic group (finite or infinite).

(ii) Construct an injective resolution of B .

The right derived functors of the functor $\text{Hom}(A, -)$ are denoted by $\text{Ext}^i(A, -)$ for $i \geq 0$. Now let A and B be cyclic groups (finite or infinite).

(iii) Compute $\text{Ext}^i(A, B)$ for all $i \geq 0$.

In the following exercises X denotes a topological space.

Exercise 5. (i) Let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on X . Show that φ is surjective if and only if for every open set U of X , and every section $s \in \mathcal{G}(U)$, there exists an open cover $(U_i)_{i \in I}$ of U and sections $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ in $\mathcal{G}(U_i)$ for all $i \in I$.

(ii) Give an example of a surjective morphism of sheaves $\varphi: \mathcal{F} \rightarrow \mathcal{G}$, and an open set U such that $\varphi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is not surjective.

Exercise 6. Let A be an abelian group. Denote by A_X the sheaf associated to the presheaf $U \mapsto A$ on X . We call A_X the *constant sheaf* on X with values in A .

(i) Let U be an open set of X . Show that $A_X(U)$ can be identified with the set of continuous maps $U \rightarrow A$. Here A is endowed with the discrete topology.

(ii) Assume that X is locally connected (i.e. for each open set U of X the connected components of U are open). Show that $A_X(U)$ can be identified with the set of locally constant functions $U \rightarrow A$. A function $U \rightarrow A$ is called locally constant if it is constant on each connected component of U .

Exercise 7. Let K be a closed subset of X , and denote by $i: K \rightarrow X$ the inclusion of K in X . Let \mathcal{F} be a sheaf on K . Denote by $i_*\mathcal{F}$ the presheaf on X given by

$$U \mapsto \mathcal{F}(K \cap U).$$

(i) Show that $i_*\mathcal{F}$ is in fact a sheaf.

(ii) Show that

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in K \\ 0 & x \notin K. \end{cases}$$

We call $i_*\mathcal{F}$ the *extension of \mathcal{F} by zero outside K* .

(iii) Show that $\mathcal{F} \mapsto i_*\mathcal{F}$ is an exact functor from Sh_K to Sh_X , i.e. sends exact sequences to exact sequences.

(iv) Show that $\mathcal{F} \mapsto i_*\mathcal{F}$ sends flasque sheaves to flasque sheaves. A sheaf \mathcal{F} is called *flasque* if for each inclusion $U \subset V$ of open sets in X , the restriction map $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ is surjective.

(v) Recall that a sheaf always admits a flasque resolution, and that this resolution can be used to compute the cohomology groups of that sheaf. Use this to show that $H^k(X, i_*\mathcal{F}) \cong H^k(K, \mathcal{F})$ for all $k \geq 0$.

Let K_1, K_2 be closed subsets of X with $X = K_1 \cup K_2$. Denote by \mathbb{Z}_i ($i = 1, 2$) the constant sheaf with values in \mathbb{Z} on K_i ($i = 1, 2$). Denote by \mathbb{Z}_{12} the constant sheaf with values in \mathbb{Z} on $K_1 \cap K_2$. Let $i_{1,2}$ be the inclusions of $K_{1,2}$ into X , and let i_{12} be the inclusion of $K_1 \cap K_2$ into X .

(vi) Show that we have a natural exact sequence of sheaves

$$0 \rightarrow \mathbb{Z}_X \rightarrow i_{1*}\mathbb{Z}_1 \oplus i_{2*}\mathbb{Z}_2 \rightarrow i_{12*}\mathbb{Z}_{12} \rightarrow 0$$

on X .

(vii) Show that by taking the long exact sequence of cohomology of this short exact sequence we get the *Mayer-Vietoris exact sequence*

$$\dots \rightarrow H^i(X, \mathbb{Z}) \rightarrow H^i(K_1, \mathbb{Z}) \oplus H^i(K_2, \mathbb{Z}) \rightarrow H^i(K_1 \cap K_2, \mathbb{Z}) \rightarrow H^{i+1}(X, \mathbb{Z}) \rightarrow \dots$$

(note that for readability we removed the subscripts from the \mathbb{Z} 's). As an intermediate step, show that sheaf cohomology commutes with taking direct sums of sheaves.