Exercises Topics in Geometry - Singular homology, Fall 2009

In the following exercises X is a topological space.

Exercise 1. Let X be a path-connected space. Prove that $\mathrm{H}_0(X) \cong \mathbb{Z}$.

Exercise 2. Let $X_1, \ldots, X_n$ be the path-connected components of the space X. Prove that $\mathrm{H}_k(X) \cong \oplus_{i=1}^{n} \mathrm{H}_k(X_i)$ for all $k \in \mathbb{Z}$.

Exercise 3. Let $\mathcal{C}'_\bullet$ be a subcomplex of a complex $\mathcal{C}_\bullet$. Let $k \in \mathbb{Z}$. Show that

$$d_{k+1}(\mathcal{C}_{k+1}) + \mathcal{C}'_k \subset d_k^{-1}(\mathcal{C}'_{k-1})$$

and that $\mathrm{H}_k(\mathcal{C}_\bullet/\mathcal{C}'_\bullet)$ can be identified with the quotient $d_k^{-1}(\mathcal{C}'_{k-1})/(d_{k+1}(\mathcal{C}_{k+1}) + \mathcal{C}'_k)$.

Exercise 4. Prove that $< e_1, e_0 > + < e_0, e_1 >$ is a boundary in $\mathcal{C}_1(\Delta^1)$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex given by $(1-t)e_0 + te_1 \mapsto s(t), t \in [0,1]$, and let $\sigma'\colon \Delta^1 \to X$ be given by $(1-t)e_0 + te_1 \mapsto s(1-t)$. Prove that $\sigma + \sigma'$ is a boundary in $\mathcal{C}_1(X)$.

Exercise 5. A singular 1-simplex $\sigma\colon \Delta^1 \to X$ is called a *loop* if $\sigma(e_0) = \sigma(e_1)$.
(a) Prove that a loop is a 1-cycle.
(b) Two loops σ_0 and σ_1 are called *freely homotopic* if there is a continuous map $F\colon [0,1] \times [0,1] \to X$ such that $F(0,t) = \sigma_0((1-t)e_0 + te_1)$ and $F(1,t) = \sigma_1((1-t)e_0 + te_1)$ and each $F(s,t)$ is a loop. Prove that free homotopy defines an equivalence relation on the set of loops in X.
(c) Prove that two freely homotopic loops are homologous.
(d) Choose a basepoint $x \in X$. Give a natural map $\rho\colon \pi_1(X,x) \to \mathrm{H}_1(X)$ and prove that it is a homomorphism. So we have a natural map $\bar{\rho}\colon \pi_1(X,x)^{\mathrm{ab}} \to \mathrm{H}_1(X)$.
(e) A 1-chain $\sigma_0 + \cdots + \sigma_{r-1}$ with $\sigma_i(e_0) = \sigma_{i-1}(e_1)$ for all $i \in \mathbb{Z}/r\mathbb{Z}$ is called an *elementary 1-cycle*. Prove that an elementary 1-cycle is a 1-cycle, homologous to a loop.
(f) Prove that the classes of loops generate $\mathrm{H}_1(X)$.
(g) Assume that X is path-connected. Show that ρ is surjective.
Remark: it can be proved that $\bar{\rho}$ is an isomorphism.

Exercise 6. Let A be a subspace of X.
(a) Assume there exists a map $r\colon X \to A$ which is the identity on A (in that case we call r a *retraction map* and A a *retract* of X). Let $k \in \mathbb{Z}$. Show that $\mathrm{H}_k(X) \cong \mathrm{H}_k(A) \oplus \operatorname{Ker} r_k$.
(b) Assume there exists a map $R\colon X \times [0,1] \to X$ such that $R(a,t) = a$ for all $a \in A$ and all t, and $R(x,0) = x$ and $R(x,1) \in A$ for all x in X (in that case we call R a *deformation retraction map* and A a *deformation retract* of X). Show that for each subspace $B \subset A$ the inclusion $(A,B) \subset (X,B)$ induces isomorphisms on homology.

Exercise 7. (a) Let $\phi\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ be a chain map of exact complexes. Suppose there exist two distinct residue classes modulo 3 such that ϕ_k is an isomorphism whenever k belongs to one of these two residue classes. Prove that ϕ_k is an isomorphism for all $k \in \mathbb{Z}$.
(b) Let $\phi\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ be a chain map of complexes. Assume $\mathcal{C}'_\bullet \subset \mathcal{C}_\bullet$ and $\mathcal{D}'_\bullet \subset \mathcal{D}_\bullet$ are sbucomplexes such that $\phi_k(\mathcal{C}'_k) \subset \mathcal{D}'_k$ for all $k \in \mathbb{Z}$. So we have chain maps $\phi'\colon \mathcal{C}'_\bullet \to \mathcal{D}'_\bullet$ and $\bar{\phi}\colon \mathcal{C}_\bullet/\mathcal{C}'_\bullet \to \mathcal{D}_\bullet/\mathcal{D}'_\bullet$. Prove that if two of $\phi, \phi'$and $\bar{\phi}$ induce an isomorphism on homology, then

so does the third.
(c) Let $f\colon (X,Y,Z) \to (X',Y',Z')$ be a map of triads. In particular we have three maps of topological pairs $(X,Y) \to (X',Y')$, $(X,Z) \to (X',Z')$ and $(Y,Z) \to (Y',Z')$. Prove that if two of these inclusions induce isomorphisms on homology, then so does the third.

Exercise 8. Let $\phi, \phi'\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ be chain maps. A *chain homotopy* from ϕ to ϕ' is a collection homomorphisms $(P_k\colon \mathcal{C}_k \to \mathcal{D}_{k+1})_{k\in\mathbb{Z}}$ such that $\phi'_k - \phi_k = P_{k-1}d_k + d_{k+1}P_k$ for all $k \in \mathbb{Z}$.
(a) Prove that chain homotopy defines an equivalence relation on the set of chain maps from $\mathcal{C}_\bullet$ to $\mathcal{D}_\bullet$.
(b) Let $\phi, \phi'\colon \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ and $\psi, \psi'\colon \mathcal{D}_\bullet \to \mathcal{E}_\bullet$ be chain homotopic. Prove that $\psi\phi, \psi'\phi'\colon \mathcal{C}_\bullet \to \mathcal{E}_\bullet$ are chain homotopic.
(c) Prove that chain homotopic maps induce the same maps on homology.

Exercise 9. The *cone* CX over a non-empty space X is obtained from $[0,1]\times X$ by identifying the subspace $\{0\} \times X$ to one point v, the *vertex* of CX.
(a) Show that CX is contractible.
Let $\{x\}$ be a one point space and let $\epsilon\colon X \to \{x\}$ be the unique map. Let $k \in \mathbb{Z}$. We define the k-th *reduced homology group* $\tilde{\mathrm{H}}_k(X)$ to be the kernel of the map $\epsilon_k\colon \mathrm{H}_k(X) \to \mathrm{H}_k(\{x\})$.
(b) Prove that $\mathrm{H}_k(CX, CX - \{v\}) \cong \tilde{\mathrm{H}}_{k-1}(X)$.

Exercise 10. Visualize the first barycentric subdivision of Δ^3 and count the number of 3-simplices in it.

Exercise 11. The *suspension* ΣX of a non-empty space X is obtained from $[0,1] \times X$ by identifying each of the subsets $\{0\} \times X$ and $\{1\} \times X$ to a point.
(a) Prove that the projection $[0,1] \times X \to [0,1]$ defines a map $h\colon \Sigma X \to [0,1]$.
(b) Compute the homology of ΣX by applying Mayer-Vietoris to the open sets $h^{-1}(0,1]$ and $h^{-1}[0,1)$.
(c) Let S^n for $n \in \mathbb{Z}_{\geq 0}$ be the n-sphere. Prove that ΣS^n and S^{n+1} are homeomorphic and compute the homology groups of S^n from this.

Exercise 12. Let $p_1, \ldots, p_n$ be distinct points in the plane $\mathbb{R}^2$. Compute the homology of $\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$.

Exercise 13. Let S^2 be the 2-sphere and let $D_1, \ldots, D_n$ be n small open discs on S^2 with disjoint boundaries. Let X^+, X^- be two copies of $S^2 \setminus (D_1 \cup \ldots \cup D_n)$ and let X be the space obtained by identifying, for each $i = 1, \ldots, n$, the boundary of D_i on X^+ with the boundary of D_i on X^-, using the identity map. Thus, X is a "sphere with $n-1$ handles". Compute the homology of X.

Exercise 14. Each graph has the homotopy type of a bouquet of circles. Suppose that X is a graph, with the homotopy type of a bouquet of n circles. Prove that n is a homotopy-invariant of X. We call n the *Betti number* of X.

Exercise 15. Suppose that X is the union of open sets $U_0, \ldots, U_n$ such that all homology groups $\mathrm{H}_k(Y)$ vanish for any intersection $Y = U_{i_0} \cap \ldots \cap U_{i_r}$ of these open sets and all $k > 0$ (we call the open cover $\{U_0, \ldots, U_n\}$ an *acyclic* cover in this case).
(a) Show that $\mathrm{H}_k(X) = 0$ for $k > n$.
(b) If, in addition, each intersection Y is path-connected or empty, and $n \geq 1$, show that $\mathrm{H}_n(X) = 0$.

Exercise 16. Let $f\colon X \to Y$ be a map between non-empty spaces.
(a) Prove that f induces a natural map $\Sigma f\colon \Sigma X \to \Sigma Y$ between the suspensions of X and Y (see Exercise 11).
(b) Let $f\colon S^n \to S^n$ be a map and let $\Sigma f\colon S^{n+1} \to S^{n+1}$ be the map induced from a homeomorphism $\Sigma S^n \cong S^{n+1}$. Prove that f and Σf have the same degree.
(c) In particular, for each $n > 0$ there exists maps $S^n \to S^n$ of arbitrary degree.

Exercise 17. In class we have seen that for any $n \geq 1$ and any $k \in \mathbb{Z}$ we have natural isomorphisms

$$\mathrm{H}_k(\Delta^n, \partial\Delta^n) \cong \mathrm{H}_{k-1}(\Delta^{n-1}, \partial\Delta^{n-1})\,.$$

Let Y be a non-empty space. By sticking in Y as a 'dummy' variable, we have natural isomorphisms

$$\mathrm{H}_k((\Delta^n, \partial\Delta^n) \times Y) \cong \mathrm{H}_{k-1}((\Delta^{n-1}, \partial\Delta^{n-1}) \times Y)$$

as well.
(a) Prove, by iteration, that $\mathrm{H}_k((\Delta^n, \partial\Delta^n) \times Y) \cong \mathrm{H}_{k-n}(Y)$.
(b) Hence we have $\mathrm{H}_k(B^n \times Y, S^{n-1} \times Y) \cong \mathrm{H}_{k-n}(Y)$.
(c) Let x be a point on S^n. Prove that $\mathrm{H}_k(S^n \times Y, \{x\} \times Y) \cong \mathrm{H}_{k-n}(Y)$.
(d) Prove that there is a natural isomorphism

$$\mathrm{H}_k(S^n \times Y) \cong \mathrm{H}_{k-n}(Y) \oplus \mathrm{H}_k(Y)\,.$$

Hint: the projection $S^n \times Y \to Y \cong \{x\} \times Y$ is a retraction.
(e) Compute the homology groups of $S^1 \times \ldots \times S^1$ (n factors).

Exercise 18. If $m, n \geq 0$ then every point z of $S^{m+n+1} \subset \mathbb{R}^{m+n+2} = \mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$ can be represented in the form $z = \cos(t) \cdot x + \sin(t) \cdot y$ with $x \in S^m$, $y \in S^n$, and $t \in [0, \pi/2]$, and this representation is unique except that x resp. y is undetermined when $t = \pi/2$ resp. $t = 0$. Given $f\colon S^m \to S^m$ and $g\colon S^n \to S^n$ we define their *join* $f * g\colon S^{m+n+1} \to S^{m+n+1}$ by $(f * g)(z) = \cos(t) \cdot f(x) + \sin(t) \cdot g(y)$.
(a) Prove that $\deg(f * g) = \deg(f) \cdot \deg(g)$. Hint: first prove that $f * g = (f * \mathrm{id})(\mathrm{id} * g)$ and prove $\deg(f * \mathrm{id}) = \deg(f)$ by induction on n. You may want to use the results of Exercise 16.
(b) Show that if both f and g are homotopic to the identity, then so is $f * g$.
(c) Exhibit a homotopy from id to $-\mathrm{id}$ on S^1.
(d) Prove that the antipodal map on an odd-dimensional sphere is homotopic to the identity.

Exercise 19. In this exercise we prove the Main Theorem of Algebra. Let $p(z) = z^k + c_1 z^{k-1} + \cdots + c_k$ with $k > 0$ be a non-constant polynomial with complex coefficients. We view S^1 as the unit circle in $\mathbb{C}$. Assume p has no zeroes. We can then define a map $\hat{p}\colon S^1 \to S^1$ via

$$\hat{p}(z) = \frac{p(z)}{|p(z)|}\,.$$

(a) Exhibit a homotopy from $\hat{p}$ to a constant map. Hint: use that p has no zero z with $|z| \leq 1$.
(b) Exhibit a homotopy from $\hat{p}$ to the map $z \mapsto z^k$. Hint: use the identity

$$t^k p(\frac{z}{t}) = z^k + t(c_1 z^{k-1} + t c_2 z^{k-2} + \cdots + t^{k-1} c_k)$$

and the fact that $\hat{p}$ has no zero z with $|z| \geq 1$.
(c) Finish the proof of the Main Theorem of Algebra.

Exercise 20. In this exercise we prove that an even-dimensional sphere cannot be given the structure of a topological group. Given a group G acting as a group of homeomorphisms of a space X, we say that G acts *freely* if the only element from G which has any fixed points is the identity element. Let g, h be two elements, unequal to the identity element, from a group G acting freely on S^n, where n is even.
(a) Prove that both g and h have degree -1.
(b) Prove that gh is the identity element.
(c) Conclude that G is either $\mathbb{Z}/2\mathbb{Z}$ or the trivial group.
(d) Prove that S^n is not a topological group.

Exercise 21. Prove that S^3 is a topological group. Hint: identify $\mathbb{R}^4$ with the Hamilton quaternions.

Exercise 22. Let $\mathbb{P}^n(\mathbb{R}) = \mathbb{P}(\mathbb{R}^{n+1})$ be the n-dimensional real projective space. Prove that any map $\mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ has a fixed point if n is even. Describe a map $\mathbb{P}^n(\mathbb{R}) \to \mathbb{P}^n(\mathbb{R})$ without fixed points for each odd n.

Exercise 23. For each $n \in \mathbb{Z}_{>0}$ construct a surjective map $S^n \to S^n$ that has degree 0.

Exercise 24. Let $n \in \mathbb{Z}_{>0}$. Prove that every map $S^n \to S^n$ is homotopic to one that has a fixed point.