Gauss map on the theta divisor and Green's functions

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Abstract

In an earlier paper we constructed a Cartier divisor on the theta divisor of a principally polarised abelian variety whose support is precisely the ramification locus of the Gauss map. In this note we discuss a Green's function associated to this locus. For jacobians we relate this Green's function to the canonical Green's function of the corresponding Riemann surface.

1 Introduction

In [7] we investigated the properties of a certain theta function η defined on the theta divisor of a principally polarised complex abelian variety (ppav for short). Let us recall its definition. Fix a positive integer g and denote by $\mathbb{H}_g$ the complex Siegel upper half space of degree g. On $\mathbb{C}^g \times \mathbb{H}_g$ we have the Riemann theta function

$$\theta = \theta(z, \tau) = \sum_{n \in \mathbb{Z}^g} \mathrm{e}^{\pi i {}^t n \tau n + 2\pi i {}^t n z} .$$

Here and henceforth, vectors are column vectors and ^{t} denotes transpose. For any fixed τ, the function $\theta = \theta(z)$ on $\mathbb{C}^g$ gives rise to an (ample, symmetric and reduced) divisor Θ on the torus $A = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ which, by this token, acquires the structure of a ppav. The theta function

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θ can be interpreted as a tautological section of the line bundle $O_A(\Theta)$ on A.

Write θ_i for the first order partial derivative $\partial\theta/\partial z_i$ and θ_{ij} for the second order partial derivative $\partial^2\theta/\partial z_i \partial z_j$. Then we define η by

$$\eta = \eta(z,\tau) = \det \left(\begin{array}{cc} \theta_{ij} & \theta_j \\ {}^t\theta_i & 0 \end{array} \right) .$$

We consider the restriction of η to the vanishing locus of θ on $\mathbb{C}^g \times \mathbb{H}_g$.

In [7] we proved that for any fixed τ the function η gives rise to a global section of the line bundle $O_\Theta(\Theta)^{\otimes g+1} \otimes \lambda^{\otimes 2}$ on Θ in $A = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$; here λ is the trivial line bundle $\mathrm{H}^0(A, \omega_A) \otimes_{\mathbb{C}} O_\Theta$, with ω_A the canonical line bundle on A. When viewed as a function of two variables (z,τ) the function η transforms as a theta function of weight $(g+5)/2$ on $\theta^{-1}(0)$. If τ is fixed then the support of η on Θ is exactly the closure in Θ of the ramification locus $\mathrm{R}(\gamma)$ of the Gauss map on the smooth locus Θ^s of Θ. Recall that the Gauss map on Θ^s is the map

$$\gamma \colon \Theta^s \longrightarrow \mathbb{P}(T_0 A)^\vee$$

sending a point x in Θ^s to the tangent space $T_x\Theta$, translated over x to a subspace of T_0A. It is well-known that the Gauss map on Θ^s is generically finite exactly when (A,Θ) is indecomposable; in particular the section η is non-zero for such ppav's.

It turns out that the form η has a rather nice application in the study of the geometry of certain codimension-2 cycles on the moduli space of ppav's. For this application we refer to the paper [5].

The purpose of the present note is to discuss a certain real-valued variant $\|\eta\| \colon \Theta \to \mathbb{R}$ of η. In the case that (A,Θ) is the jacobian of a Riemann surface X we will establish a relation between this $\|\eta\|$ and the canonical Green's function of X. In brief, note that in the case of a jacobian of a Riemann surface X we can identify Θ^s with the set of effective divisors of degree $g-1$ on X that do not move in a linear system; thus for such divisors D it makes sense to define $\|\eta\|(D)$. On the other hand, note that Θ^s carries a canonical involution σ coming from the action of -1 on A, and moreover note that sense can be made of evaluating the canonical (exponential) Green's function G of X on pairs of effective divisors of X. The relation that we shall prove is then of the form

$$\|\eta\|(D) = \mathrm{e}^{-\zeta(D)} \cdot G(D, \sigma(D)) \, ;$$

here D runs through the divisors in Θ^s, and ζ is a certain continuous function on Θ^s. The ζ from the above formula is intimately connected with the geometry of intersections $\Theta \cap (\Theta + R - S)$, where R, S are distinct points on X. Amusingly, the limits of such intersections where R and S approach each other are hyperplane sections of the Gauss map corresponding to points on the canonical image of X, so the Gauss map on the theta divisor is connected with the above formula in at least two different ways.

2 Real-valued variant of η

Let $(A = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g), \Theta = \operatorname{div}\theta)$ be a ppav as in the introduction. As we said, the function η transforms like a theta function of weight $(g+5)/2$ and order $g+1$ on Θ. This implies that if we define

$$\|\eta\| = \|\eta\|(z,\tau) = (\det Y)^{(g+5)/4} \cdot \mathrm{e}^{-\pi(g+1)^t y \cdot Y^{-1} \cdot y} \cdot |\eta(z,\tau)| \, ,$$

where $Y = \operatorname{Im}\tau$ and $y = \operatorname{Im} z$, we obtain a (real-valued) function which is invariant for the action of Igusa's transformation group $\Gamma_{1,2}$ of matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{Sp}(2g,\mathbb{Z})$ with a, b, c, d square matrices such that the diagonals of both ${}^t ac$ and ${}^t bd$ consist of even integers. Recall that $\Gamma_{1,2}$ acts on $\mathbb{C}^g \times \mathbb{H}_g$ via

$$(z,\tau) \mapsto \left({}^t(c\tau + d)^{-1} z, (a\tau + b)(c\tau + d)^{-1}\right) \, .$$

It follows that $\|\eta\|$ is a well-defined function on Θ, equivariant with respect to isomorphisms $(A,\Theta) \xrightarrow{\sim} (A',\Theta')$ coming from the symplectic action of $\Gamma_{1,2}$ on $\mathbb{H}_g$. Note that the zero locus of $\|\eta\|$ on Θ coincides with the zero locus of η on Θ. In fact, if (A,Θ) is indecomposable then the function $-\log\|\eta\|$ is a Green's function on Θ associated to the closure of $\mathrm{R}(\gamma)$.

The definition of $\|\eta\|$ is a variant upon the definition of the function

$$\|\theta\| = \|\theta\|(z,\tau) = (\det Y)^{1/4} \cdot \mathrm{e}^{-\pi^t y \cdot Y^{-1} \cdot y} \cdot |\theta(z,\tau)|$$

that one finds in [4], p. 401. We note that $\|\theta\|$ should be seen as the norm of θ for a canonical hermitian metric $\|\cdot\|_{\mathrm{Th}}$ on $O_A(\Theta)$; we obtain $\|\eta\|$ as the norm of η for the induced metric on $O_\Theta(\Theta)^{\otimes g+1} \otimes \lambda^{\otimes 2}$. Here $\mathrm{H}^0(A,\omega_A)$ has the standard metric given by putting $\|\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_g\| = (\det Y)^{1/2}$.

The curvature form of $(O_A(\Theta), \|\cdot\|_{\mathrm{Th}})$ on A is the translation-invariant

$(1,1)$-form

$$\mu = \frac{\mathrm{i}}{2} \sum_{k=1}^{g} \mathrm{d}z_k \wedge \overline{\mathrm{d}z_k} \,.$$

The (g,g)-form $\frac{1}{g!}\mu^g$ is a Haar measure for A giving A measure 1. As μ represents Θ we have

$$\frac{1}{g!} \int_\Theta \mu^{g-1} = 1 \,.$$

If (A,Θ) is indecomposable then $\log \|\eta\|$ is integrable with respect to μ^{g-1} and the integral

$$\frac{1}{g!} \int_\Theta \log \|\eta\| \cdot \mu^{g-1}$$

is a natural real-valued invariant of (A,Θ). We come back to it in Remark 4.6 below.

3 Arakelov theory of Riemann surfaces

The purpose of this section and the next is to investigate the function $\|\eta\|$ in more detail for jacobians. There turns out to be a natural connection with certain real-valued invariants occurring in the Arakelov theory of Riemann surfaces. We begin by recalling the basic notions from this theory [1] [4].

Let X be a compact and connected Riemann surface of positive genus g, fixed from now on. Denote by ω_X its canonical line bundle. On $\mathrm{H}^0(X,\omega_X)$ we have a natural inner product $(\omega,\eta) \mapsto \frac{\mathrm{i}}{2}\int_X \omega \wedge \overline{\eta}$; we fix an orthonormal basis $(\omega_1,\ldots,\omega_g)$ with respect to this inner product.

We put

$$\nu = \frac{\mathrm{i}}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k} \,.$$

This is a $(1,1)$-form on X, independent of our choice of $(\omega_1,\ldots,\omega_g)$ and hence canonical. In fact, if one denotes by (J,Θ) the jacobian of X and by $j \colon X \hookrightarrow J$ an embedding of X into J using line integration, then $\nu = \frac{1}{g} j^*\mu$ where μ is the translation-invariant form on J discussed in the previous section. Note that $\int_X \nu = 1$.

The canonical Green's function G of X is the unique non-negative function $X \times X \to \mathbb{R}$ which is non-zero outside the diagonal and satisfies

$$\frac{1}{\mathrm{i}\pi}\partial\overline{\partial} \log G(P,\cdot) = \nu(P) - \delta_P \,, \qquad \int_X \log G(P,Q)\nu(Q) = 0$$

for each P on X; here δ denotes Dirac measure. The functions $G(P,\cdot)$ give rise to canonical hermitian metrics on the line bundles $O_X(P)$, with curvature form equal to ν.

From G, a smooth hermitian metric $\|\cdot\|_{\mathrm{Ar}}$ can be put on ω_X by declaring that for each P on X, the residue isomorphism

$$\omega_X(P)[P] = (\omega_X \otimes_{O_X} O_X(P))[P] \xrightarrow{\sim} \mathbb{C}$$

is an isometry. Concretely this means that if $z : U \to \mathbb{C}$ is a local coordinate around P on X then

$$\|\mathrm{d}z\|_{\mathrm{Ar}}(P) = \lim_{Q \to P} |z(P) - z(Q)|/G(P,Q)\,.$$

The curvature form of the metric $\|\cdot\|_{\mathrm{Ar}}$ on ω_X is equal to $(2g-2)\nu$.

We conclude with the delta-invariant of X. Write $J = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ and $\Theta = \operatorname{div}\theta$. There is a standard and canonical identification of (J,Θ) with $(\mathrm{Pic}_{g-1}X, \Theta_0)$ where $\mathrm{Pic}_{g-1}X$ is the set of linear equivalence classes of divisors of degree $g-1$ on X, and where $\Theta_0 \subseteq \mathrm{Pic}_{g-1}X$ is the subset of $\mathrm{Pic}_{g-1}X$ consisting of the classes of effective divisors. By the identification $(J,\Theta) \cong (\mathrm{Pic}_{g-1}X, \Theta_0)$ the function $\|\theta\|$ can be interpreted as a function on $\mathrm{Pic}_{g-1}X$.

Now recall that the curvature form of $(O_J(\Theta), \|\cdot\|_{\mathrm{Th}})$ is equal to μ. This boils down to an equality of currents

$$\frac{1}{\mathrm{i}\pi}\partial\overline{\partial}\log\|\theta\| = \mu - \delta_\Theta$$

on J. On the other hand one has for generic $P_1,\ldots,P_g$ on X that $\|\theta\|(P_1 + \cdots + P_g - Q)$ vanishes precisely when Q is one of the points P_k. This implies that on X the equality of currents

$$\frac{1}{\mathrm{i}\pi}\partial_Q\overline{\partial}_Q \log\|\theta\|(P_1 + \cdots + P_g - Q) = j^*\mu - \sum_{k=1}^{g}\delta_{P_k} = g\nu - \sum_{k=1}^{g}\delta_{P_k}$$

holds. Since also

$$\frac{1}{\mathrm{i}\pi}\partial_Q\overline{\partial}_Q \log\prod_{k=1}^{g} G(P_k,Q) = g\nu - \sum_{k=1}^{g}\delta_{P_k}$$

we may conclude, by compactness of X, that

$$\|\theta\|(P_1 + \cdots + P_g - Q) = c(P_1,\ldots,P_g)\cdot\prod_{k=1}^{g} G(P_k,Q)$$

for some constant $c(P_1,\ldots,P_g)$ depending only on $P_1,\ldots,P_g$. A closer analysis (cf. [4], p. 402) reveals that

$$c(P_1, \ldots, P_g) = \mathrm{e}^{-\delta/8} \cdot \frac{\| \det \omega_i(P_j) \|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)}$$

for some constant δ which is then by definition the delta-invariant of X. The argument to prove this equality uses certain metrised line bundles and their curvature forms on sufficiently big powers X^r of X. A variant of this argument occurs in the proof of our main result below.

4 Main result

In order to state our result, we need some more notation and facts. We still have our fixed Riemann surface X of positive genus g and its jacobian (J, Θ). The following lemma is well-known.

Lemma 4.1. *Under the identification $\Theta \cong \Theta_0$, the smooth locus Θ^s of Θ corresponds to the subset Θ_0^s of Θ_0 of divisors that do not move in a linear system. Furthermore, there is a tautological surjection Σ from the $(g-1)$-fold symmetric power $X^{(g-1)}$ of X onto Θ_0. This map Σ is an isomorphism over Θ_0^s.*

The lemma gives rise to identifications $\Theta^s \cong \Theta_0^s \cong \Upsilon$ with Υ a certain open subset of $X^{(g-1)}$. We fix and accept these identifications in all that follows. Note that the set Υ carries a canonical involution σ, coming from the action of -1 on J. For D in Υ the divisor $D + \sigma(D)$ of degree $2g-2$ is always a canonical divisor.

The next lemma gives a description of the ramification locus of the Gauss map on $\Theta^s \cong \Upsilon$.

Lemma 4.2. *Under the identification $\Theta^s \cong \Upsilon$ the ramification locus of the Gauss map on Θ^s corresponds to the set of divisors D in Υ such that D and $\sigma(D)$ have a point in common.*

Proof. According to [3], p. 691 the ramification locus of the Gauss map is given by the set of divisors $E + P$ with E effective of degree $g-2$ and P a point such that on the canonical image of X the divisor $E + 2P$ is contained in a hyperplane. But this condition on E and P means that $E + 2P$ is dominated by a canonical divisor, or equivalently, that P is contained in the conjugate $\sigma(E+P)$ of $E+P$. The lemma follows. □

If $D = P_1 + \cdots + P_m$ and $D' = Q_1 + \cdots + Q_n$ are two effective divisors on X we define $G(D,D')$ to be

$$G(D,D') = \prod_{i=1}^{m} \prod_{j=1}^{n} G(P_i, Q_j) \,.$$

Clearly the value $G(D,D')$ is zero if and only if D and D' have a point in common. Applying this to the above lemma, we see that the function $D \mapsto G(D, \sigma(D))$ on Υ vanishes precisely on the ramification locus of the Gauss map. As a consequence $G(D,\sigma(D))$ and $\|\eta\|(D)$ have exactly the same zero locus. It looks therefore as if a relation

$$\|\eta\|(D) = \mathrm{e}^{-\zeta(D)} \cdot G(D, \sigma(D))$$

should hold for D on Υ with ζ a suitable continuous function. The aim of the rest of this note is to prove this relation, and to compute ζ explicitly.

We start with

Proposition 4.3. *Let $Y = \Upsilon \times X \times X$. The map $\|\Lambda\| \colon Y \to \mathbb{R}$ given by*

$$\|\Lambda\|(D,R,S) = \frac{\|\theta\|(D+R-S)}{G(R,S)G(D,S)G(\sigma(D),R)}$$

is continuous and nowhere vanishing. Furthermore $\|\Lambda\|$ factors via the projection of Y onto Υ.

Proof. The numerator $\|\theta\|(D+R-S)$ vanishes if and only if $R=S$ or $D = E+S$ for some effective divisor E of degree $g-2$ or $D+R$ is linearly equivalent to an effective divisor E' of degree g such that $E' = E'' + S$ for some effective divisor E'' of degree $g-1$. The latter condition is precisely fulfilled when the linear system $|D+R|$ is positive dimensional, or equivalently, by Riemann-Roch, when $D+R$ is dominated by a canonical divisor, i.e. when R is contained in $\sigma(D)$. It follows that the numerator $\|\theta\|(D+R-S)$ and the denominator $G(R,S)G(D,S)G(\sigma(D),R)$ have the same zero locus on Y. Fixing a divisor D in Υ and using what we have said in Section 3 it is seen that the currents

$$\frac{1}{\mathrm{i}\pi}\partial\overline{\partial} \log \|\theta\|(D+R-S) \text{ and } \frac{1}{\mathrm{i}\pi}\partial\overline{\partial} \log \left(G(R,S)G(D,S)G(\sigma(D),R)\right)$$

are both the same on $X \times X$. We conclude that $\|\Lambda\|$ is non-zero and continuous and depends only on D. $\square$

We also write $\|\Lambda\|$ for the induced map on Υ. Our main result is

Theorem 4.4. *Let D be an effective divisor of degree $g-1$ on X, not moving in a linear system. Then the formula*

$$\|\eta\|(D) = \mathrm{e}^{-\delta/4} \cdot \|\Lambda\|(D)^{g-1} \cdot G(D, \sigma(D))$$

holds.

Proof. Fix two distinct points R, S on X. We start by proving that there is a non-zero constant c depending only on X such that

$$(*) \quad \|\eta\|(D) = c \cdot G(D, \sigma(D)) \left(\frac{\|\theta\|(D+R-S)}{G(R,S)G(D,S)G(\sigma(D),R)} \right)^{g-1}$$

for all D varying through Υ. We would be done if we could show that

$$\frac{1}{\mathrm{i}\pi} \partial\overline{\partial} \log \|\eta\|(D)$$

and

$$\frac{1}{\mathrm{i}\pi} \partial\overline{\partial} \log \left(G(D, \sigma(D)) \left(\frac{\|\theta\|(D+R-S)}{G(R,S)G(D,S)G(\sigma(D),R)} \right)^{g-1} \right)$$

define the same currents on Υ. Indeed, then the function $\phi(D)$ given by

$$\log \|\eta\|(D) - \log \left(G(D, \sigma(D)) \left(\frac{\|\theta\|(D+R-S)}{G(R,S)G(D,S)G(\sigma(D),R)} \right)^{g-1} \right)$$

is pluriharmonic on Υ, hence on Θ^s, and since Θ^s is open in Θ with boundary empty or of codimension ≥ 2, and since Θ is normal (cf. [8], Theorem 1') we may conclude that ϕ is constant.

To prove equality of

$$\frac{1}{\mathrm{i}\pi} \partial\overline{\partial} \log \|\eta\|(D)$$

and

$$\frac{1}{\mathrm{i}\pi} \partial\overline{\partial} \log \left(G(D, \sigma(D)) \left(\frac{\|\theta\|(D+R-S)}{G(R,S)G(D,S)G(\sigma(D),R)} \right)^{g-1} \right)$$

on Υ it suffices to prove that their pullbacks are equal on $\Upsilon' = p^{-1}(\Upsilon)$ in X^{g-1} under the canonical projection $p : X^{g-1} \to X^{(g-1)}$.

First of all we compute the pullback under p of

$$\frac{1}{\mathrm{i}\pi} \partial\overline{\partial} \log \|\eta\|(D)$$

on Υ'. Let $\pi_i \colon X^{g-1} \to X$ for $i = 1, \ldots, g-1$ be the projections onto the various factors. We have seen that the curvature form of $O_J(\Theta)$ is

μ, hence the curvature form of $O_\Theta(\Theta)^{\otimes g+1}$ is $(g+1)\mu_\Theta$. According to [4], p. 397 the pullback of μ_Θ to X^{g-1} under the canonical surjection $\Sigma\colon X^{g-1} \to \Theta$ can be written as

$$\frac{\mathrm{i}}{2}\sum_{k=1}^{g}\left(\sum_{i=1}^{g-1}\pi_i^*(\omega_k)\right)\wedge\left(\sum_{i=1}^{g-1}\pi_i^*(\overline{\omega_k})\right) .$$

Here $(\omega_1, \ldots, \omega_g)$ is an orthonormal basis for $\mathrm{H}^0(X, \omega_X)$ which we fix. Let's call the above form ξ. It follows that

$$p^* \frac{1}{\mathrm{i}\pi}\partial\overline{\partial}\log\|\eta\|(D) = (g+1)\xi - \delta_{p^*\mathrm{R}(\gamma)}$$

as currents on Υ'. Here $\mathrm{R}(\gamma)$ is the ramification locus of the Gauss map on Υ.

Next we consider the pullback under p of

$$\frac{1}{\mathrm{i}\pi}\partial\overline{\partial}\log\left(G(D,\sigma(D))\left(\frac{\|\theta\|(D+R-S)}{G(R,S)G(D,S)G(\sigma(D),R)}\right)^{g-1}\right)$$

on Υ'. The factor $\|\theta\|(D+R-S)$ accounts for a contribution equal to ξ, and both of the factors $G(D,S)$ and $G(\sigma(D),R)$ give a contribution $\sum_{i=1}^{g-1}\pi_i^*(\nu)$. We find

$$p^*\frac{1}{\mathrm{i}\pi}\partial\overline{\partial}\log\left(\frac{\|\theta\|(D+R-S)}{G(R,S)G(D,S)G(\sigma(D),R)}\right)^{g-1} = (g-1)(\xi - 2\sum_{i=1}^{g-1}\pi_i^*(\nu)) .$$

We are done if we can prove that

$$p^*\frac{1}{\mathrm{i}\pi}\partial\overline{\partial}\log G(D,\sigma(D)) = 2\xi + (2g-2)\sum_{i=1}^{g-1}\pi_i^*(\nu) - \delta_{p^*\mathrm{R}(\gamma)} .$$

For this consider the product $\Upsilon' \times \Upsilon' \subseteq X^{g-1} \times X^{g-1}$. For $i,j = 1,\ldots,g-1$ denote by $\pi_{ij} : X^{g-1}\times X^{g-1} \to X \times X$ the projection onto the i-th factor of the left X^{g-1}, and onto the j-th factor of the right X^{g-1}. Denoting by Φ the smooth form represented by $\frac{1}{\mathrm{i}\pi}\partial\overline{\partial}\log G(P,Q)$ on $X \times X$ it is easily seen that we can write

$$p^*\frac{1}{\mathrm{i}\pi}\partial\overline{\partial}\log G(D,\sigma(D)) + \delta_{p^*\mathrm{R}(\gamma)} = (\sigma^*\sum_{i,j=1}^{g-1}\pi_{ij}^*\Phi)|_\Delta ;$$

here $\Delta \cong \Upsilon'$ is the diagonal in $\Upsilon' \times \Upsilon'$ and σ^* is the action on symmetric $(1,1)$-forms on Υ' induced by the automorphism $(x,y) \mapsto (x,\sigma(y))$ of

$\Upsilon \times \Upsilon$. Let q_1, q_2 be the projections of $X \times X$ onto the first and second factor, respectively. Then according to [1], Proposition 3.1 we have

$$\begin{aligned} \Phi &= \frac{\mathrm{i}}{2g}\sum_{k=1}^{g} q_1^*(\omega_k)\wedge q_1^*(\overline{\omega_k}) + \frac{\mathrm{i}}{2g}\sum_{k=1}^{g} q_2^*(\omega_k)\wedge q_2^*(\overline{\omega_k}) \\ &\quad -\frac{\mathrm{i}}{2}\sum_{k=1}^{g} q_1^*(\omega_k)\wedge q_2^*(\overline{\omega_k}) - \frac{\mathrm{i}}{2}\sum_{k=1}^{g} q_2^*(\omega_k)\wedge q_1^*(\overline{\omega_k})\,. \end{aligned}$$

Note that $q_1 \cdot \pi_{ij} = \pi_i$ and $q_2 \cdot \pi_{ij} = \pi_j$; this gives

$$\begin{aligned} \pi_{ij}^*\Phi &= \frac{\mathrm{i}}{2g}\sum_{k=1}^{g} \pi_i^*(\omega_k)\wedge \pi_i^*(\overline{\omega_k}) + \frac{\mathrm{i}}{2g}\sum_{k=1}^{g} \pi_j^*(\omega_k)\wedge \pi_j^*(\overline{\omega_k}) \\ &\quad -\frac{\mathrm{i}}{2}\sum_{k=1}^{g} \pi_i^*(\omega_k)\wedge \pi_j^*(\overline{\omega_k}) - \frac{\mathrm{i}}{2}\sum_{k=1}^{g} \pi_j^*(\omega_k)\wedge \pi_i^*(\overline{\omega_k})\,. \end{aligned}$$

Next note that σ acts as -1 on $\mathrm{H}^0(X, \omega_X)$; this implies, at least formally, that

$$\begin{aligned} \sigma^*\pi_{ij}^*\Phi &= \frac{\mathrm{i}}{2g}\sum_{k=1}^{g} \pi_i^*(\omega_k)\wedge \pi_i^*(\overline{\omega_k}) + \frac{\mathrm{i}}{2g}\sum_{k=1}^{g} \pi_j^*(\omega_k)\wedge \pi_j^*(\overline{\omega_k}) \\ &\quad +\frac{\mathrm{i}}{2}\sum_{k=1}^{g} \pi_i^*(\omega_k)\wedge \pi_j^*(\overline{\omega_k}) + \frac{\mathrm{i}}{2}\sum_{k=1}^{g} \pi_j^*(\omega_k)\wedge \pi_i^*(\overline{\omega_k}) \\ &= \pi_i^*(\nu) + \pi_j^*(\nu) + \frac{\mathrm{i}}{2}\sum_{k=1}^{g} \pi_i^*(\omega_k)\wedge \pi_j^*(\overline{\omega_k}) \\ &\quad +\frac{\mathrm{i}}{2}\sum_{k=1}^{g} \pi_j^*(\omega_k)\wedge \pi_i^*(\overline{\omega_k})\,. \end{aligned}$$

We obtain for $(\sigma^* \sum_{i,j=1}^{g-1} \pi_{ij}^*\Phi)|_\Delta$ the expression

$$\begin{aligned} &\sum_{i,j=1}^{g-1} \pi_i^*(\nu) + \pi_j^*(\nu) + \frac{\mathrm{i}}{2}\sum_{k=1}^{g}\sum_{i,j=1}^{g-1} \pi_i^*(\omega_k)\wedge \pi_j^*(\overline{\omega_k}) + \pi_j^*(\omega_k)\wedge \pi_i^*(\overline{\omega_k}) \\ &\qquad = (2g-2)\sum_{i=1}^{g-1}\pi_i^*(\nu) + \mathrm{i}\sum_{k=1}^{g}\left((\sum_{i=1}^{g-1}\pi_i^*(\omega_k)) \wedge (\sum_{i=1}^{g-1}\pi_i^*(\overline{\omega_k}))\right) \\ &\qquad = (2g-2)\sum_{i=1}^{g-1}\pi_i^*(\nu) + 2\xi\,, \end{aligned}$$

and this gives us what we want.

It remains to prove that the constant c is equal to $\mathrm{e}^{-\delta/4}$. We use the following lemma.

Lemma 4.5. *Let* $\mathrm{Wr}(\omega_1, \ldots, \omega_g)$ *be the Wronskian on* $(\omega_1, \ldots, \omega_g)$, *considered as a global section of* $\omega_X^{\otimes g(g+1)/2}$. *Let* P *be any point on* X. *Then the equality*

$$\|\eta\|((g-1)P) = \mathrm{e}^{-(g+1)\delta/8} \cdot \|\mathrm{Wr}(\omega_1, \ldots, \omega_g)\|_{\mathrm{Ar}}(P)^{g-1}$$

holds. Left and right hand side are non-vanishing for generic P.

Proof. Let $\kappa \colon X \to \Theta$ be the map given by sending P on X to the linear equivalence class of $(g-1) \cdot P$. According to [6], Lemma 3.2 we have a canonical isomorphism

$$\kappa^*(O_\Theta(\Theta)) \otimes \omega_X^{\otimes g} \xrightarrow{\sim} \omega_X^{\otimes g(g+1)/2} \otimes \kappa^*(\lambda)^{\otimes -1}$$

of norm $\mathrm{e}^{\delta/8}$. It follows that we have a canonical isomorphism

$$\kappa^* \left(O_\Theta(\Theta)^{\otimes g+1} \otimes \lambda^{\otimes 2} \right) \xrightarrow{\sim} \left(\omega_X^{\otimes g(g+1)/2} \otimes \kappa^*(\lambda)^{\otimes -1} \right)^{\otimes g-1}$$

of norm $\mathrm{e}^{(g+1)\delta/8}$. Chasing these isomorphisms using [7], Theorem 5.1 one sees that the global section $\kappa^*\eta$ of

$$\kappa^* \left(O_\Theta(\Theta)^{\otimes g+1} \otimes \lambda^{\otimes 2} \right)$$

is sent to the global section

$$\left(\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\omega_1 \wedge \ldots \wedge \omega_g} \cdot \mathrm{Wr}(\omega_1, \ldots, \omega_g) \right)^{\otimes g-1}$$

of

$$\left(\omega_X^{\otimes g(g+1)/2} \otimes \kappa^*(\lambda)^{\otimes -1} \right)^{\otimes g-1} .$$

The claimed equality follows. The non-vanishing for generic P follows from $\mathrm{Wr}(\omega_1, \ldots, \omega_g)$ being non-zero as a section of $\omega_X^{\otimes g(g+1)/2}$. □

We can now finish the proof of Theorem 4.4. Using the defining relation

$$\|\theta\|(P_1 + \cdots + P_g - S) = \mathrm{e}^{-\delta/8} \frac{\|\det \omega_i(P_j)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \prod_{k=1}^{g} G(P_k, S)$$

mentioned earlier for δ we can rewrite equality (*) as

$$\|\eta\|(D) = c \cdot \mathrm{e}^{-(g-1)\delta/8} \frac{G(D, \sigma(D))}{G(R, \sigma(D))^{g-1}} \left(\frac{\|\det \omega_i(P_j)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \right)^{g-1} ;$$

here we have set $D = P_1 + \cdots + P_{g-1}$ and $P_g = R$. Letting the P_j approach R we find, by a similar computation as in [6], proof of Lemma 3.2,

$$\|\eta\|((g-1)R) = c \cdot \mathrm{e}^{-(g-1)\delta/8} \cdot \|\mathrm{Wr}(\omega_1, \ldots, \omega_g)\|_{\mathrm{Ar}}(R)^{g-1} .$$

Lemma 4.5 gives $c \cdot \mathrm{e}^{-(g-1)\delta/8} = \mathrm{e}^{-(g+1)\delta/8}$, in other words $c = \mathrm{e}^{-\delta/4}$. □

Remark 4.6. It was shown by J.-B. Bost [2] that there is an invariant A of X such that for each pair of distinct points R, S on X the formula

$$\log G(R,S) = \frac{1}{g!} \int_{\Theta + R - S} \log \|\theta\| \cdot \mu^{g-1} + A$$

holds. An inspection of the proof as for example given in [9], Section 5 reveals that the integrals

$$\frac{1}{g!} \int_{\Theta^s} \log G(D,S) \cdot \mu(D)^{g-1} \quad \text{and} \quad \frac{1}{g!} \int_{\Theta^s} \log G(\sigma(D),R) \cdot \mu(D)^{g-1}$$

are zero and hence from the definition of $\|\Lambda\|$ we can write

$$A = -\frac{1}{g!} \int_{\Theta^s} \log \|\Lambda\|(D) \cdot \mu(D)^{g-1} .$$

A combination with the formula in Theorem 4.4 yields

$$-\frac{1}{g!} \int_{\Theta} \log \|\eta\| \cdot \mu^{g-1} = \frac{\delta}{4} + (g-1)A - \frac{1}{g!} \int_{\Theta^s} \log G(D, \sigma(D)) \cdot \mu(D)^{g-1} .$$

This formula might be considered interesting since the left hand side is an invariant of ppav's, whereas the right hand side is only defined for Riemann surfaces.

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