Compact representations of quadratic numbers

The Pell equation $x^2 - dy^2 = 1$, for a given non-square positive integer d, is one of the most basic Diophantine equations, and it appears in a wide variety of problems.

The Pell equation is always solvable, and all the solutions $x + y\sqrt{d}$ can be written as $x + y\sqrt{d} = (x_1 + y_1\sqrt{d})^n, n \in \mathbb{Z}$, where $x_1 + y_1\sqrt{d}$ is called the *fundamental solution.*

The fundamental solution can always be found using continued fractions, but if d is large this method becomes slow. The reason for this is that the solution itself is huge. To make faster algorithms for solving the Pell equation it is neccessary to write down the solution in a different way.

A *compact representation* of β is a product of the form

$$\beta = \prod_{j=1}^{k} \alpha_j^{2^j}, \tag{1}$$

where $\alpha_j = \frac{a_j + b_j\sqrt{d}}{c_j}$, $a_j, b_j, c_j \in \mathbb{Z}$, and none of the α_j can be too large.

The key to obtaining a compact representation of the solution of the Pell equation is to exploit the connection of the fundamental solution of the Pell equation with the fundamental unit η_d of the ring of integers of $\mathbb{Q}(\sqrt{d})$. Note that compact representations enable modern algorithms to solve the Pell equation in less time than it actually takes to write down the solution in standard representation!

The goal of this project is to explain the notion of the *infrastructure* of a real quadratic field and to explain how one constructs a compact representation of η_d.

References

[1] M. J. Jacobson Jr., H. C. Williams, *Solving the Pell Equation*, Springer, 2009.