Rational points on varieties, part II (surfaces)

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1. Picard group and canonical divisor

- Criterion for φ_L being a closed immersion in terms of linear system L [5, Remark II.7.8.2], [6, Theorem A3.2.1].
- Ampleness on curves [5, Proposition IV.3.1 and Corollary IV.3.2], [6, A3.2.2].
- Kodaira dimension [5, Section V.6], [6, Section F5.1].
- Kodaira dimension for curves [6, Example F5.1.1].
- Classification of surfaces [5, Section V.6], [6, F5.1].
- General type or very canonical [5, Section V.6], [6, F5.2], [10, Section I.2].
- Bombieri–Lang conjecture [6, Section F5.2], [10, Section I.3].
- Batyrev–Manin conjectures [2, 15], [6, Section F5.4].
- Heuristics for number of rational points on surface of degree d in $\mathbb{P}^3$.

2. Exercises

(1) Suppose $X \subset \mathbb{P}^n$ is a projective variety (so irreducible) over k of positive dimension. Denote the inclusion $X \subset \mathbb{P}^n$ by ι. Let $H' \in \operatorname{Div} \mathbb{P}^n$ be a hyperplane that does not contain X, and set $H = \iota^*(H')$ (so H is a hyperplane section on X).

   (a) We let $\mathcal{L}_{\mathbb{P}^n}(H')$ and $\mathcal{L}_X(H)$ denote the usual k-subspaces of the function fields $k(\mathbb{P}^n)$ and $k(X)$, respectively. Show that restriction yields a natural linear map
   $$\iota^* \colon \mathcal{L}_{\mathbb{P}^n}(H') \to \mathcal{L}_X(H).$$
   (b) Show by examples that the linear map ι^* of (1) need not be injective and ι^* need not be surjective, either. [Hint: for the failure of surjectivity, search through some examples of [5, Section II.7].]

   (c) Show that for every integer m we have
   $$\kappa(X, mH) = \begin{cases} -\infty & (m < 0), \\ 0 & (m = 0), \\ \dim X & (m > 0). \end{cases}$$

(2) Find all sequences $n, e_1, e_2, \ldots, e_t$ of integers at least 2, such that complete intersections in $\mathbb{P}^n$ of hypersurfaces of degrees $e_1, e_2, \ldots, e_t$ are not of general type and have dimension at most 3.

(3) Let $f \colon \mathbb{P}^1(r, s) \times \mathbb{P}^1(t, u) \to \mathbb{P}^3(x, y, z, w)$ be the Segre embedding, given by
$$([r : s], [t : u]) \mapsto [rt : ru : st : su].$$
   (a) Which linear system L on $\mathbb{P}^1 \times \mathbb{P}^1$ is this map associated to?

   (b) Show that the image X of f is given by $yz = xw$ and show that f induces an isomorphism onto its image.

   (c) Find an $a \in \mathbb{Q}$ such that aH is a canonical divisor on X for every hyperplane section H of X.

   (d) Show that there exists a constant c such that
   $$\#\{P \in X(\mathbb{Q}) \;:\; H_{\mathbb{P}^3}(P) \leq B\} \sim cB^2 \log B$$
   as B goes to ∞.

3. Next time

- Extended moving lemma and intersection numbers constant within divisor classes [6, A2.3.1].
- Intersection pairing on $\operatorname{Pic} X$ when X is normal and projective surface [5, Theorem V.1.1], [6, Section A2.3], [7, Appendix B].

- Self intersection: $C \cdot D = \deg_C \mathcal{L}(D) \otimes \mathcal{O}_C$ restricted to $C = D$ [5, Lemma V.1.3].
- $X \subset \mathbb{P}^n$ a surface, $H \in \operatorname{Div} X$ a hyperplane section, $C \subset X$ a curve. Then $H^2 = H \cdot H = \deg X$ [6, A2.3], and $H \cdot C = \deg C$. [5, Exercise V.1.2].
- Adjunction formula $2g(C) - 2 = C \cdot (C + K_X)$ for smooth curve C on smooth projective surface X [5, Proposition V.1.5], [6, Theorem A4.6.2].
- Riemann-Roch for surfaces [5, Theorem V.1.6], [6, Theorem A4.6.3].
- Kodaira Vanishing [5, Remark II.7.15, Exercise V.4.12], [6, Remark A4.6.3.2].
- Let $f : S \to S'$ be a surjective morphism of smooth, irreducible, projective surfaces that is generically finite of degree d. Then for any $D, D' \in \operatorname{Div} S'$, we have $(f^*D) \cdot (f^*D') = d(D \cdot D')$ [3, Proposition I.8] for characteristic zero, [6, A2.3.2] for f finite, combine [11, Propositions 5.2.32 and 9.2.11] for the general case.
- Blow-up [3, Section II.1], [5, Section I.4], [6, A1.2.6.(f)].
  - effect on Pic
  - effect on canonical divisor
  - self intersection
  - numerical conditions for being an exceptional curve
  - self intersection of strict transforms
- algebraic and numerical equivalence
  - difference is torsion
  - when all the same?
- ample + (divisor with no base points) = ample
- cubic surfaces
  - embedding
  - 27 lines
- del Pezzo surfaces

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