Rational points on varieties, part II (surfaces)

Ronald van Luijk

WONDER, November 21, 2013

1. Del Pezzo surfaces and Brauer-Severi varieties

- Del Pezzo surfaces, including classification over separably closed fields [22].
- Cubic surfaces [5, Section V.4], [13, Chapter IV], [22].
- Kodaira Vanishing Theorem for rational surfaces over an algebraically closed field of positive characteristic.
- Segre-Manin Theorem [13, Theorem 29.4], [22].
- Brauer-Severi varieties with a rational point are trivial [22].

2. Exercises

(1) For geometrically rational surfaces, Kodaira's vanishing theorem also holds in characteristic p: let X be a geometrically rational surface with canonical divisor K_X and let D be an ample divisor. Then we have $s(D + K_X) = 0$.
   (a) Let X be a del Pezzo surface of degree d. Show that for all positive integers m we have $\ell(-mK_X) = 1 + \frac{1}{2}m(m+1)d$.
   (b) Suppose $d = 4$. Show that X is isomorphic with the complete intersection of two quadric surfaces in $\mathbb{P}^4$.

(2) Take your favorite field k and your favorite 6-tuple of points $P_1, \ldots, P_6 \in \mathbb{P}^2_k$ in general position. Let X be the blow up of $\mathbb{P}^2$ in these six points. As we have seen, the linear system $|-K_X|$ induces an embedding of X into $\mathbb{P}^3$. Compute (with computer, probably) an equation of the image.

(3) Let $\pi\colon X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ in r points $P_1, P_2, \ldots, P_r$. For each i, let $E_i \subset X$ denote the exceptional curve above P_i.
   (a) Use exercise 1 from last week to show that if $C \subset \mathbb{P}^2$ is a nice curve of degree d, and $\tilde{C} \subset X$ is its strict transform, then on X we have $\tilde{C}^2 = d^2 - m$, where m is the number of points among $P_1, \ldots, P_r$ that lie on C.
   (b) Conclude that the strict transform of a line through exactly two points and the strict transform of a smooth conic through exactly five points are exceptional curves on X. Note that for $r = 6$, together with $E_1, \ldots, E_6$, this accounts for all 27 exceptional curves on X.
   (c) For each $r \in \{1, \ldots, 8\}$, find the number of exceptional curves on X, and describe their images in $\mathbb{P}^2$, assuming the points are in general position.

(4) Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the "Cremona transformation", given by

$$[x : y : z] \mapsto [yz : xz : xy].$$

   (a) Show that φ is not well defined at the points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$, and $P_3 = [0 : 0 : 1]$, but that φ^2 extends to the identity.
   (b) Let $\pi\colon X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points P_1, P_2, P_3. Show that φ extends to an automorphism of X in the sense that there exists an automorphism $\tilde{\varphi}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^2 \arrow[r, "\varphi"'] & \mathbb{P}^2 \end{tikzcd}$$

   commutative.

(5) Pascal's Theorem states the following. Let $P_1, \ldots, P_6$ be six points on an irreducible conic $\Gamma \subset \mathbb{P}^2$. Let Q, R, and S be the three intersection points of the lines P_1P_2 and P_4P_5, the lines P_2P_3 and P_5P_6, and the lines P_3P_4 and P_6P_1, respectively. Then Q, R, and S are collinear. Prove this theorem.

References

[1] M. Atiyah and I. MacDonald, Introduction to commutative algebra, Addison-Wesley, 1969.

[2] V. Batyrev and Yu. Manin, Sur le nombre des points rationnels de hauteur borné des variétés algébriques, Math. Ann. **286** (1990), no. 1-3, 27–43.

[3] A. Beauville, Complex algebraic surfaces, second edition, LMS Student texts **34**, Cambridge University Press, 1996.

[4] D. Eisenbud, Commutative algebra, with a view toward algebraic geometry, Graduate Texts in Mathematics **150**, corrected third printing, Springer, 1999.

[5] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics **52**, corrected eighth printing, Springer, 1997.

[6] M. Hindry and J. Silverman, Diophantine Geometry. An Introduction, Graduate Texts in Mathematics, **201**, Springer, 2000.

[7] S.L. Kleiman, *The Picard scheme*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. **123**, Amer. Math. Soc., Providence, RI, 2005, 235-321.

[8] J. Kollár, *Unirationality of cubic hypersurfaces*, J. Inst. Math. Jussieu **1** (2002), no. 3, 467–476.

[9] S. Lang, Algebra, third edition, Addison-Wesley, 1997.

[10] S. Lang, Survey of Diophantine geometry, second printing, Springer, 1997.

[11] Q. Liu, Algebraic geometry and arithmetic curves, translated by Reinie Erné, Oxford GTM **6**, 2002.

[12] R. van Luijk, *Density of rational points on elliptic surfaces*, Acta Arithmetica, Volume **156** (2012), no. 2, 189–199.

[13] Yu. Manin, *Cubic Forms*, North-Holland, 1986.

[14] H. Matsumura, Commutative algebra, W.A. Benjamin Co., New York, 1970.

[15] E. Peyre, Counting points on varieties using universal torsors, Arithmetic of higher dimensional algebraic varieties, eds. B. Poonen and Yu. Tschinkel, Progress in Mathematics **226**, Birkhäuser, 2003.

[16] M. Pieropan, *On the unirationality of Del Pezzo surfaces over an arbitrary field*, Algant Master thesis, `http://www.algant.eu/documents/theses/pieropan.pdf`.

[17] B. Poonen, Rational points on varieties, `http://www-math.mit.edu/~poonen/papers/Qpoints.pdf`

[18] B. Poonen and Yu. Tschinkel, Arithmetic of higher dimensional algebraic varieties, Progress in Mathematics **226**, Birkhäuser, 2003.

[19] B. Segre, *A note on arithmetical properties of cubic surfaces*, J. London Math. Soc. **18** (1943), 24–31.

[20] B. Segre, *On the rational solutions of homogeneous cubic equations in four variables*, Math. Notae **11** (1951), 1–68.

[21] Sir P. Swinnerton-Dyer, *Diophantine equations: progress and problems*, Arithmetic of higher dimensional algebraic varieties, eds. B. Poonen and Yu. Tschinkel, Progress in Mathematics **226**, Birkhäuser, 2003.

[22] A. Várilly-Alvarado, *Arithmetic of del Pezzo and K3 surfaces*, `http://math.rice.edu/~av15/dPsK3s.html`.